Chapter 9

Basic on the simplex method

The Simplex Method is certainly the most famous and most used algorithm in optimization. Proposed in 1947 by G.B.Dantzig, he has undergone, in over 50 years of life, many improvements which, while not changing substantially the simple logic structure created by Dantzig, have certainly improved the computational efficiency and ease of use. Today there are many commercial " packages " that implement the Simplex algorithm and allow the solution of Linear Programming problems with millions of variables.

9.1 Introduction

The Simplex Method applies to linear programming problems in *standard form* :

$$\begin{array}{ll} \min & c^T x \\ & Ax = b \geq 0 \\ & x \geq 0 \end{array} \tag{9.1}$$

with $b \geq 0$.
Based on the fundamental theorem for LP, the simplex method looks for a it vertex of

$$P = \{x \in \Re^n : Ax = b, \quad x \geq 0_n\}$$

which is optimal for problem (9.1).

We briefly describe an implementation of the method of Simplex in it two phases: it Phase I and it Phase II.

Phase I allows to check whether the LP problem is feasible and, in case, to find an initial vertex.

Theorem 8.8 guarantees the existence of a vertex for a polyhedron in (9.1). If the problem does not have a vertex then the polyhedron P is empty and the problem is unfeasible and Phase I ends.

Phase I consists in

- check of feasibility;

- elimination redundant constraints so that $\text{rank}(A) = m$;
- find a feasible vertex.

Starting from this first solution, the simplex method (Phase II) produces a movement along the edges of the feasible polyhedron, so to pass from one vertex to an adjacent one improving the value of the objective function to find an optimal solution of problem (P) or to conclude that the problem is unbounded.

Phase II solves a problem of type (9.1) with $\text{rank}(A) = m$ starting form a first feasible vertex. The algorithm produces a sequence of feasible vertexes, checking at each iteration either optimality of the solution or unboundednes of the problem using suitable criteria.

The main steps are:

1. optimality certification of the current vertex;
2. unboundedness certification;
3. construction of a new feasible vertex.

Under suitable assupmtions, the simplex method converges in a finite number of iterations to an optimal solution of 9.1, or certifies that such a solution does not exist.

The algorithm used in Phase II can be used to solve Phase I too. we report a short description of the main concepts useful to understand the output of most commercial software, starting from the Phase II of the simplex method. Phase I will be described later

9.1.1 Basic Feasible Solution (BFS) and the reduced problem

A submatrix B ($m \times m$) of A is said the be a *basic matrix* of A if it is not singular. Given a basis B, we can express A as $A = (B,\ N)$, after reordering of the columns, where N is the submatrix obtained with the column not in B. Analogously we partion the ectors

$$x = \begin{pmatrix} x_B \\ x_N \end{pmatrix}, \qquad c = \begin{pmatrix} c_B \\ c_N \end{pmatrix}$$

Variables x_B are the *basic variables* whereas x_N *not basic variables*. We can write the problem as:

$$\begin{aligned} &\min c_B^T x_B + c_N^T x_N \\ &Bx_B + Nx_N = b \\ &x_B \geq 0, \\ &x_N \geq 0 \end{aligned}$$

Since B is non singular we have

$$x_B = B^{-1}b - B^{-1}Nx_N \tag{9.2}$$

Substituting back we get a problem in the only variables x_N.

The *reduced problem* in the only variables x_N

$$\begin{array}{ll} \min & c_B^T B^{-1} b + (c_N^T - c_B^T B^{-1} N) x_N \\ & B^{-1} b - B^{-1} N x_N \geq 0_m, \\ & x_N \geq 0 \end{array} \tag{9.3}$$

is equivalent to problem (9.1)

In particular, a vector $\hat{x} = \begin{pmatrix} \hat{x}_B \\ \hat{x}_N \end{pmatrix}$ is a feasible solution of (9.1) if and only if $\hat{x}_N$ is feasible for (9.3) e $\hat{x}_B = B^{-1}b - B^{-1}N\hat{x}_N$ and the value of the objective function of problem (9.1) in $\hat{x}$ is equal to the objective value of the reduced problem in $\hat{x}_N$.

Given a basis B, a feasible solution $x_B = B^{-1}b$, $x_N = 0$ is called *Basic feasible Solution* (BFS) if and only if $B^{-1}b \geq 0$.

Theorem 9.2 *A feasible point $\bar{x}$ is a vertex of problem* (9.1) *if and only if is BFS.*

Hence given the BFS $\bar{x} = \begin{pmatrix} B^{-1}b \\ 0_{n-m} \end{pmatrix}$ asscoited to the basis B, the point $\bar{x}_N = 0_{n-m}$ is the corresponding solution of tge reducd problem and $c^T B^{-1} b$ is the value of the objective function.

The correspondence (vertex - basis) is not unique. Given a basis you can associate a BFS, but given a BFS, i.e. a vertex, you can have more than one basis associated.

The coefficients of x_N in the objective function of the reduced problem are the components of the vector γ defined by:

$$^T\gamma = ^T c_N - ^T c_B B^{-1} N$$

called *reduced costs*.

Theorem 9.3 (Optimality criterion) *Let $\bar{x} = \begin{pmatrix} B^{-1}b \\ 0_{n-m} \end{pmatrix}$ be a BFS for problem* (9.1). If *the reduced cost is non negative, namely:*

$$^T c_N - ^T c_B B^{-1} N \geq ^T 0_{n-m},$$

then *the BFS $\bar{x} = \begin{pmatrix} B^{-1}b \\ 0_{n-m} \end{pmatrix}$ is an optimal solution for problem* (9.1).

Proof. We prove that if $\gamma \geq 0$ then $c^T x \geq c^T \bar{x}$ for all feasible x. We have $c^T \bar{x} = c_B^T \bar{x}_B + c_N^T \bar{x}_N = c_B^T \bar{x}_B = c_B^T B^{-1} b$. In any feasible solution $x = \begin{pmatrix} x_B \\ x_N \end{pmatrix}$ of (9.1) we have $c^T x = c_B^T x_B + c_N^T x_N$ and using (9.2) we get

$$c^T x = c_B{}^T B^{-1} b + \gamma^T x_N$$

Since x is feasible we have that $x_N \geq 0$, and since $\gamma \geq 0$: we get the result

$$c^T x \geq c_B{}^T B^{-1} b = c^T \bar{x}.$$

Esempio 7 *Consider the LP*

$$\begin{array}{ll} \min & x_1 + 3x_2 + x_3 \\ & x_1 + 5x_2 + 2x_3 = 6 \\ & 2x_1 + x_2 - x_4 = 2 \\ & x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0\ x_4 \geq 0 \end{array}$$

The BFS can be found by enumeration.

1. $x_1 = x_2 = 0$ *(unfeasible)*

2. $x_1 = x_3 = 0$ *(unfeasible)*

3. $x_1 = x_4 = 0$ *(unfeasible)*

4. $x_2 = x_3 = 0, x_1 = 6, x_4 = 10$ *(BFS)*

5. $x_2 = x_4 = 0, x_1 = 1, x_3 = \frac{5}{2}$ *(BFS)*

6. $x_3 = x_4 = 0, x_1 = \frac{4}{9}\ x_2 = \frac{10}{9}$ *(BFS).*

Consider the BFS $\begin{pmatrix} 6 \\ 0 \\ 0 \\ 10 \end{pmatrix}$*; the basic variable are* $x_B = \begin{pmatrix} x_1 \\ x_4 \end{pmatrix} = \begin{pmatrix} 6 \\ 10 \end{pmatrix}$*, the non basic variables are* $x_N = \begin{pmatrix} x_2 \\ x_3 \end{pmatrix} = 0$*; the basis* $B = \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}$*. The reduced cost* $\gamma^T = c_N^T - c_B^T B^{-1} N$ *and we get* $B^{-1} = \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}$*. We get*

$$\gamma^T = \begin{pmatrix} -2 & -1 \end{pmatrix} \not\geq 0,$$

we cannot conlulde on the optimality of the point $\bar{x}$*.*

9.3.1 Phase I: Construction of the first BFS

As noted in the introduction to this section, the procedure that determines the BFS for a linear programming problem is called *Phase I* of the Simplex Method. The purpose of Phase I is to determine whether the problem (9.1) is feasible and in case it identifies the first BFS. In this Phase the method also verify that $rank(A) = m$ and in case redundant constraints are removed.

Phase I is based on the definition of the *auxiliary problem*

$$\begin{array}{ll} \min & z(\alpha, x) = \sum_{i=1}^{m} \alpha_i \\ & Ax + I_m \alpha = b \\ & x \geq 0_n, \alpha \geq 0_m \end{array} \tag{9.4}$$

where $\alpha^T = (\alpha_1, \ldots, \alpha_m)$ are called *auxiliary variables*.

We note that

- the auxiliary problem has always a vertex.
- The matrix $(A \ \ I_m)$ has rank equal to m.
- A BFS dor problem (9.4) is easily identied as $(x, \alpha) = (0, b)$.
- Problem (9.4) is not unbounded below

Hence there exists always am optimal solution (x^*, α^*).

Theorem 9.4 *Problem* (9.1) *has a feasible solution if and only if the optimal solution* $\begin{pmatrix} \alpha^* \\ x^* \end{pmatrix}$ *of problem* (9.4) *has value* $z(\alpha^*, x^*) = 0$.

The solution (x^*, α^*) of (9.4) can be found using Phase II of the simplex starting from the initial BFS $(x, \alpha) = (0, b)$. If $z(x^*, \alpha^*) \neq 0$ problem is unfeasible. Otherwise at the end f Phase, the methos returns also a BFS for (9.1).