OPERATIONS RESEARCH

EXAM January 14, 2019

IMPORTANT: READ CAREFULLY

The grade on the written exam is valid at most for three months.

Check the score of each exercise.

Please note that YES NO answers are NOT valid if you do not give an explanation.

SURNAME **NAME**

I **authorize** the publication (paper and electronic) of the results obtained in the examination in accordance with the Italian Law 675/96 and subsequent amendments

SIGNATURE

GRADE

I solved the following exercise

1. ☐ **(Score 7)**
2. ☐ **(Score 5)**
3. ☐ **(Score 8)**
4. ☐ **(Score 6)**
5. ☐ **(Score 6)**

ORAL PART: questions

GRADE

Exercise 1. **(Score 7)** Consider the following nonlinear programming problem

$$\begin{aligned}\min_{x\in I\!R^3}\quad & x_1^2+x_2^2+\frac{1}{2}x_3^2+x_1x_3+3x_2x_3+2x_1+3x_3\\ & -x_1+x_2+x_3=1\\ & -3x_1-x_2+x_3\le -3\\ & x_1,x_2,x_3\ge 0\end{aligned}$$

Unconstrained problem (score 2)

(i) **(score 1)** Consider the unconstrained problem (remove the constraints). Is there any unconstrained stationary point ? Which kind of point is it ?

(ii) **(score 1)** Consider the point $x^0 = (1,\ 0,\ 4)^T$ and write the first iteration of the gradient method with exact line search to obtain the new point x^1

Constrained problem (score 5)

(iii) **(score 1)** State if the problems is convex or strictly convex or none of the two.

(iv) **(score 2)** Consider the point $\widehat{x} = (1,\ 2,\ 0)^T$ and write the KKT conditions in $\widehat{x}$. Evaluate the multipliers. Are the KKT conditions satisfied ?

(v) **(score 1)** Write the system to get a feasible and descent direction in $\widehat{x}$. Does a solution of the system (namely a feasible and descent direction) exist?

(vi) **(score 1)** Consider the point $x^0 = (1,\ 2,\ 0)^T$ and write the linear problem to find the direction of the conditional gradient (Frank-Wolfe) algorithm.

Exercise 2. **(Score 4)** Consider the following Linear programming problem

$$\begin{aligned}\min\quad & 7x_1+7x_2+\tfrac{9}{2}x_3\\ & -x_1+x_2+x_3=1\\ & -3x_1-x_2+x_3\le -3\\ & x_1,x_2,x_3\ge 0\end{aligned}$$

(i) **(score 1,5)** Find a a feasible direction along which it is possible to move from $\widehat{x} = (1,\ 2,\ 0)^T$ finding an additional active constraint. Find the stepsize $t^{\max}$ and the corresponding new point y. Is the direction also a descent one?

(ii) **(score 0,5)** Write the problem in the standard form for the simplex method.

(iii) **(score 1)** Check if the following point are vertex ot not and justify the answer.

$$\begin{pmatrix}\frac{1}{2}\\ \frac{3}{2}\\ 0\end{pmatrix}\quad\begin{pmatrix}1\\ 3\\ 1\end{pmatrix}\quad\begin{pmatrix}1\\ 2\\ 0\end{pmatrix}\quad\begin{pmatrix}0\\ 3\\ -2\end{pmatrix}$$

(iv) **(score 1)** Write a BFS for the problems at point (ii) and specify the corresponding matrices B and N.

Exercise 3. **(Score 8)** Consider the following Linear programming problem

$$\begin{aligned}\min\quad & 7x_1+7x_2+\tfrac{9}{2}x_3\\ & -x_1+x_2+x_3=1\\ & -3x_1-x_2+x_3\le -3\\ & x_1,x_2,x_3\ge 0\end{aligned}$$

(i) (score 1.5) Write the dual problem

(ii) (score 2) Solve graphically the dual: plot the feasible region, the level lines of the objective function, identify graphically the solution and find its value.

(iii) (score 2) Using duality theory, state if the primal problem has an optimal solution and in the affirmative case find it.

(iv) (score 1) State how the value of the optimal solution change if the rhs of the second constraint change from -3 to $-3+\varepsilon$ with $\varepsilon > 0$ and sufficiently small.

(v) (score 1.5) State which is the maximum value of $\varepsilon > 0$ for which the analysis in point (iv) holds.

Exercise 4. (Score 6) Consider the following integer linear programming problem

$$\begin{array}{ll} \max & x_1 + 3x_2 \\ & -x_1 + 3x_2 \leq 7 \\ & x_1 + x_2 \leq 7 \\ & x_1 - x_2 \leq \dfrac{9}{2} \\ & x_1, x_2 \geq 0 \\ & x \text{integer} \end{array}$$

Let $(1,\ 0)^T$ be a feasible point.

(i) (score 1) Find a lower and an upper bound (specify how you find them).

(ii) (score 2) Write the two subproblem $\mathcal{P}_1, \mathcal{P}_2$ obtained by branching with respect to the fractional variable x_1

(iii) (score 1.5) Solve problem $\mathcal{P}_1$. Explain if it is possible to close it.

(iv) (score 1.5) Solve problem $\mathcal{P}_2$. Explain if it is possible to close it.

Exercise 5. (Score 6)

A bank makes four kinds of loans to its personal customers and these loans yield the following annual interest rates to the bank:

- First mortgage 14%
- Second mortgage 20%
- Home improvement 20%
- Personal overdraft 10%

The bank has a maximum foreseeable lending capability of € 250 million and is further constrained by the policies:

- first mortgages must be at least 55% of all mortgages issued and at least 25% of all loans issued (in € terms)
- second mortgages cannot exceed 25% of all loans issued (in € terms)
- to avoid public displeasure and the introduction of a new windfall tax the average interest rate on all loans must not exceed 15%.

Formulate the bank's loan problem as an LP so as to maximise interest income whilst satisfying the policy limitations.