OPERATIONS RESEARCH

EXAM December 22, 2016

Exercise 1. (Score 7) Consider the following nonlinear programming problem

$$\min \quad (x_1 - 5x_2)^2 + (x_2 - 2)^2$$

$$\begin{aligned}
& x_1 - x_2 \leq 4 \\
& 7x_1 - 3x_2 \geq -21 \\
& 11x_1 + 7x_2 \leq 77 \\
& x_1 + 2x_2 \geq 2 \\
& x_1 \geq 0
\end{aligned}$$

(i) (score 1) Consider the unconstrained problem (remove the constraints) and find the unconstrained stationary points, if any. Analyze which type of points they are (minimizer, saddle, maximizer).

(ii) (score 1) Are the KKT conditions necessary and sufficient for optimality ?

(iii) (score 2) Consider the point $\widehat{x} = (0,\ 1)^T$. Write the KKT conditions in $\widehat{x}$. Are the KKT conditions satisfied (evaluate the multipliers) ?

(iv) (score 1) Write the system to get a feasible and descent direction in $\widehat{x} = (0,\ 1)^T$. Is there a solution to the system ?

(v) (score 2) Find a a feasible direction along which it is possible to move from $\tilde{x} = (4,\ 0)^T$ finding an additional active constraint. Find the stepsize $t^{\max}$ and the corresponding new point. Is the direction also a descent one?

Exercise 2. (Score 3) Consider the following Linear programming problem

$$\min \qquad 4x_1 + 21x_2 + 77x_3 - 2x_4$$

$$\begin{aligned}
x_1 - 7x_2 + 11x_3 - x_4 &\geq 1 \\
-x_1 + 3x_2 + 7x_3 - 2x_4 &= 0 \\
x_1, x_2, x_3, x_4 &\geq 0
\end{aligned}$$

(i) (score 1) Is the point $(\frac{7}{18},\ 0,\ \frac{1}{18},\ 0)^T$ a vertex of the feasible polyedron ?

(ii) (score 1) Write the problem in the standard form for the simplex method. ($Ax = b,\ x \geq 0$ with $b \geq 0$).

(iii) (score 1) Write a BFS for the problems at point (ii) and specify the corresponding matrices B and N.

Exercise 3. (Score 7) Consider the following Linear programming problem

$$\min \qquad 4x_1 + 21x_2 + 77x_3 - 2x_4$$

$$\begin{aligned}
x_1 - 7x_2 + 11x_3 - x_4 &\geq 1 \\
-x_1 + 3x_2 + 7x_3 - 2x_4 &= 0 \\
x_1, x_2, x_3, x_4 &\geq 0
\end{aligned}$$

(i) **(score 2)** Write the dual problem

(ii) **(score 2)** Solve graphically the dual: plot the feasible region, the level lines of the objective function, identify graphically the solution and find its value.

(iii) **(score 2)** Using duality theory, state if the primal problem has an optimal solution and in the affirmative case find it.

(iv) **(score 1)** State how the value of the optimal solution change if the rhs of the first constraint change from 1 to $1+\varepsilon$ with $\varepsilon > 0$ and sufficiently small.

Exercise 4. **(score 5)** Consider the following Multiobjective Linear Programming problem

$$\begin{aligned} \max \quad & \{x_1, -3x_1 + x_2\} \\ & x_1 - x_2 \leq 4 \\ & 7x_1 - 3x_2 \geq -21 \\ & 11x_1 + 7x_2 \leq 77 \\ & x_1 + 2x_2 \geq 2 \\ & x_1 \geq 0 \end{aligned}$$

(i) **(score 1)** Find the two optimal point (x^{*1}, x^{*2}) of each of the two objectives and the corresponding ideal vector (z^{*1}, z^{*2})

(ii) **(score 2)** Plot the imagine of the feasible region on the Objective function space and find the Pareto front. Which are the Pareto points ?

(iii) **(score 2)** Write the KKT conditions. Are the KKT satisfied in the point $\hat{x} = (\frac{5}{2},\ 0)^T$? Explain your answer

Exercise 5. **(Score 4)** Consider the following integer linear programming problem

$$\begin{aligned} \max \quad & x_1 \\ & x_1 - x_2 \leq 4 \\ & 7x_1 - 3x_2 \geq -21 \\ & 11x_1 + 7x_2 \leq 77 \\ & x_1 + 2x_2 \geq 2 \\ & x_1 \geq 0 \\ & x_i \in Z,\ i = 1, \dots, 2 \end{aligned}$$

Let $(2,\ 0)^T$ be a feasible point.

(i) **(score 1)** Write the lower and upper bounds at the first iteration of a Branch and Bound method.

(ii) **(score 1)** Write the two subproblem $\mathcal{P}_1, \mathcal{P}_2$ obtained by branching with respect to variable x_2

(iii) **(score 1)** Is it possible to close $\mathcal{P}_1$? Explain your answer

(iv) **(score 1)** Is it possible to close $\mathcal{P}_2$? Explain your answer

Exercise 6. (Score 5)

The Auto Company of Italy (ACI) produces four types of cars: subcompact, compact, intermediate, and luxury. ACI also produces trucks and vans. Vendor capacities limit total production capacity to at most 1 200 000 vehicles per year. Subcompacts and compacts are built together in a facility with a total annual capacity of 620 000 cars. Intermediate and luxury cars are produced in another facility with capacity of 400 000; and the truck/van facility has a capacity of 275 000. ACIs marketing strategy requires that subcompacts and compacts must constitute at least half of the product mix for the four car types. Profit margins, market potential, and fuel efficiencies (Kilometers Per Liters) are summarized below.

Type	Profit margin (€/vehicle)	Potential sales (in 000s)	Fuel efficiency (KPL)
Subcompact	150	600	40
Compact	225	400	34
Intermediate	250	300	15
Luxury	500	225	12
Truck	400	325	20
Van	200	100	25

The Corporate Average Fuel Efficiency (CAFE) standards require an average fleet fuel efficiency of at least 27 KPL. ACI would like to use a linear programming model for planning its production so to maximize the profit.