OPERATIONS RESEARCH

EXAM July 19, 2017

IMPORTANT: READ CAREFULLY

The grade on the written exam is valid at most for three months.

Check the score of each exercise.

Report your answers to the exercises in the appropriate boxes below.

You must include the sheets with your calculus and explanations that will be used to verify the correct procedure.

Calculation errors that lead to a wrong answer may not be taken into account.

Please note that $\boxed{\text{YES}}$ $\boxed{\text{NO}}$ answers are NOT valid if you do not give an explanation.

Exercise 1. **(Score 7)** Consider the following nonlinear programming problem

$$
\begin{array}{ll}
\min_{x \in I\!R^3} & x_1^3 + 2x_1^2 + 3x_2^2 + 2x_1x_2 \\
 & \\
 & x_1 - 5x_2 \leq -2 \\
 & 2x_1 + 2x_2 \leq 1 \\
 & -6x_1 + 2x_2 \leq 9 \\
 & x_2 \geq 0
\end{array}
$$

(i) (score 1) Consider the unconstrained problem (remove the constraints). Is there any unconstrained stationary point ? State if they are local minima/maxima or saddle points?

(ii) (score 1) Are the KKT conditions necessary and sufficient for optimality ?

(iii) (score 2) Consider the point $\widehat{x} = (\frac{1}{12},\ \frac{5}{12})^T$ and write the system to get a feasible and descent direction in $\widehat{x}$. Does a solution of the system (namely a feasible and descent direction) exist?

(iv) (score 1) Write the KKT conditions in $\widehat{x}$. Are the KKT conditions satisfied ? Evaluate the multipliers

(v) (score 2) Find a a feasible direction along which it is possible to move from $\tilde{x} = (-\frac{1}{2},\ 1)^T$ finding an additional active constraint. Find the stepsize $t^{\max}$ and the corresponding new point y. Is the direction also a descent one?

Exercise 2. **(Score 3)** Consider the following Linear programming problem

$$
\begin{array}{ll}
\min & -2x_1 + x_2 + 9x_3 \\
 & x_1 + 2x_2 - 6x_3 = 5 \\
 & 5x_1 - 2x_2 - 2x_3 \leq 3 \\
 & x_1, x_2, x_3 \geq 0
\end{array}
$$

(i) (score 1) Is the point $(1,\ 2,\ 0)^T$ a vertex of the feasible polyedron ?

(ii) **(score 1)** Write the problem in the standard form for the simplex method. ($Ax = b,\ x \geq 0$ with $b \geq 0$).

(iii) **(score 1)** Write a BFS for the problems at point (ii) and specify the corresponding matrices B and N.

Exercise 3. **(Score 7)** Consider the following Linear programming problem

$$\begin{array}{ll} \min & -2x_1 + x_2 + 9x_3 \\ & x_1 + 2x_2 - 6x_3 = 5 \\ & 5x_1 - 2x_2 - 2x_3 \leq 3 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

(i) **(score 2)** Write the dual problem

(ii) **(score 2)** Solve graphically the dual: plot the feasible region, the level lines of the objective function, identify graphically the solution and find its value.

(iii) **(score 2)** Using duality theory, state if the primal problem has an optimal solution and in the affirmative case find it.

(iv) **(score 1)** State how the value of the optimal solution change if the rhs of the second constraint change from 3 to $3 + \varepsilon$ with $\varepsilon > 0$ and sufficiently small.

Exercise 4. **(score 5)** Consider the following Multiobjective Linear Programming problem

$$\begin{array}{ll} \max & \{-5x_1 + 3x_2, x_1 - 2x_2\} \\ & x_1 - 5x_2 \leq -2 \\ & 2x_1 + 2x_2 \leq 1 \\ & -6x_1 + 2x_2 \leq 9 \\ & x_2 \geq 0 \end{array}$$

(i) **(score 1)** Find the two optimal point (x^{*1}, x^{*2}) of each of the two objectives and the corresponding ideal vector (z^{*1}, z^{*2})

(ii) **(score 2)** Plot the imagine of the feasible region into the Objective function space. Find the Pareto front and plot them in the graph.

(iii) **(score 2)** Write the KKT conditions. Are the KKT satisfied in the point $\hat{x} = (-1,\ \frac{1}{2})^T$? Justify your answer by evaluating the multipliers.

Exercise 5. **(Score 4)** Consider the following integer linear programming problem

$$\begin{array}{ll} \max & -5x_1 + 3x_2 \\ & x_1 - 5x_2 \leq -2 \\ & 2x_1 + 2x_2 \leq 1 \\ & -6x_1 + 2x_2 \leq 9 \\ & x_2 \geq 0 \\ & x \text{integer} \end{array}$$

Let $(-1,\ 1)^T$ be a feasible point.

(i) **(score 1)** Solve the linear relaxation and find the upper bound to the optimal integer solution.

(ii) **(score 1)** Write the two subproblem $\mathcal{P}_1, \mathcal{P}_2$ obtained by branching with respect to fractional variable x_2

(iii) **(score 1)** Solve problem $\mathcal{P}_1$. Explain if it is possible to close it.

(iv) **(score 1)** Solve problem $\mathcal{P}_2$. Explain if it is possible to close it.

Exercise 6. (Score 5)

Brown Furniture Company makes three kinds of office furniture P_1, P_2, and P_3: chairs, desks, and tables. Each product requires skilled labor in the parts fabrication department, unskilled labor in the assembly department, machining on some key pieces of equipment, and some wood as raw material.

The unit profit for each product are r_1, r_2, and r_3, and the company can sell everything that it manufactures. Each product P_i need a processing time t_{ij} in the fabrication department, in the assembly department and in the machining equipment. The number of skilled and unskilled available labor hours are S, U and T is the available time on the relevant equipment. A given quantity W of wood can be bought. The data are summarized in the table below.

	Chairs	Desks	Tables	Resources available
Fabrication (hr)	4	6	2	S =2000 hr
Assembly (hr)	3	8	6	U =2000 hr
Machining (hr)	9	6	4	T =1440 hr
Wood (sq. ft)	30	40	25	W = 9600 sq. ft
Profit per unit	€16	€20	€14	

Write an LP problem to solve the production problem of Managers at Brown Furniture.