OPERATIONS RESEARCH - TEXT A

EXAM July 20, 2018

IMPORTANT: READ CAREFULLY

The grade on the written exam is valid at most for three months.

Check the score of each exercise.

Please note that YES NO answers are NOT valid if you do not give an explanation.

SURNAME **NAME**

I **authorize** the publication (paper and electronic) of the results obtained in the examination in accordance with the Italian Law 675/96 and subsequent amendments

SIGNATURE

GRADE

I solved the following exercise

1. ☐ **(Score 6)**
2. ☐ **(Score 5)**
3. ☐ **(Score 8)**
4. ☐ **(Score 6)**
5. ☐ **(Score 6)**

ORAL PART: questions

GRADE

Exercise 1. (Score 6) Consider the following nonlinear programming problem

$$\begin{aligned} \min_{x\in\mathbb{R}^2} \quad & \tfrac{1}{2}(x_1 + 2x_1x_2 + x_2)^2 + x_2 \\ & 2x_1 + x_2 \geq -1 \\ & -3x_1 + 2x_2 \leq \tfrac{4}{3} \\ & x_1 \leq \tfrac{5}{6} \\ & x_2 \leq \tfrac{7}{2} \\ & x_2 \geq 0 \end{aligned}$$

Unconstrained problem (score 2)

(i) **(score 1)** Consider the point $x^0 = (1, \; -1)^T$ and write the first iteration of the gradient method with exact line search to obtain the new point x^1

(ii) **(score 1)** Consider the unconstrained problem (remove the constraints). Find the unconstrained stationary point and determine the nature of the points.

Constrained problem (score 4)

(iii) **(score 1)** State if the problems is convex or strictly convex or none of the two.

(iv) **(score 2)** Consider the point $\widehat{x} = (0, \; \frac{2}{3})^T$ and write the KKT conditions in $\widehat{x}$. Are the KKT conditions satisfied ? Evaluate the multipliers

(v) **(score 1)** Write the system to get a feasible and descent direction in $\widehat{x}$. Does a solution of the system (namely a feasible and descent direction) exist?

Exercise 2. (Score 5) Consider the following Linear programming problem

$$\begin{aligned} \min \quad & x_1 + \tfrac{4}{3}x_2 + \tfrac{7}{2}x_3 + \tfrac{5}{3}x_4 \\ & -2x_1 - 3x_2 + 2x_4 = 2 \\ & x_1 - 2x_2 - x_3 \leq -1 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

(i) **(score 2)** Find a a feasible direction along which it is possible to move from $\widehat{x} = (\frac{1}{4}, \; 0, \; 4, \; \frac{5}{4})^T$ finding an additional active constraint. Find the stepsize $t^{\max}$ and the corresponding new point y. Is the direction also a descent one?

(ii) **(score 0,5)** Write the problem in the standard form for the simplex method.

(iii) **(score 0,5)** Write the auxiliary problem to be solved in Phase I of the simplex method.

(iv) **(score 1)** Check if the following points are vertex

$$\begin{pmatrix} 1 \\ 0 \\ 3 \\ 2 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 2 \\ 0 \\ 4 \end{pmatrix} \quad \begin{pmatrix} \frac{1}{2} \\ 0 \\ 4 \\ \frac{3}{2} \end{pmatrix} \quad \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

(v) **(score 1,25)** Write a BFS for the problems at point (ii) and specify the corresponding matrices B and N.

Exercise 3. (Score 8) Consider the following Linear programming problem

$$\begin{aligned} \min \quad & x_1 + \tfrac{4}{3}x_2 + \tfrac{7}{2}x_3 + \tfrac{5}{3}x_4 \\ & -2x_1 - 3x_2 + 2x_4 = 2 \\ & x_1 - 2x_2 - x_3 \leq -1 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

(i) **(score 1.5)** Write the dual problem

(ii) **(score 2)** Solve graphically the dual: plot the feasible region, the level lines of the objective function, identify graphically the solution and find its value.

(iii) **(score 2)** Using duality theory, state if the primal problem has an optimal solution and in the affirmative case find it.

(iv) **(score 1)** State how the value of the optimal solution change if the rhs of the second constraint change from -1 to $-1 \pm \varepsilon$ with $\varepsilon > 0$ and sufficiently small.

(v) **(score 1.5)** State which is the maximum value of $\varepsilon > 0$ for which the analysis in point (iv) holds.

Exercise 4. (Score 6) Consider the following integer linear programming problem

$$\begin{aligned} \max \quad & 2x_1 + x_2 \\ & 2x_1 + x_2 \geq -1 \\ & -3x_1 + 2x_2 \leq \tfrac{4}{3} \\ & x_1 \leq \tfrac{5}{6} \\ & x_2 \leq \tfrac{7}{2} \\ & x_2 \geq 0 \\ & x \text{ integer} \end{aligned}$$

Let $(0,\ 0)^T$ be a feasible point.

(i) **(score 1)** Report a lower and an upper bound to the optimal value of the objective function.

(ii) **(score 1)** Write the two subproblem $\mathcal{P}_1, \mathcal{P}_2$ obtained by branching with respect to the fractional variable x_2

(iii) **(score 2)** Solve problem $\mathcal{P}_1$. Explain if it is possible to update the incumbent and/or to close it (you can use the graphical solution).

(iv) **(score 2)** Solve problem $\mathcal{P}_2$. Explain if it is possible to update the incumbent and/or to close it (you can use the graphical solution).

Exercise 5. (Score 6)

A company produces coal in $M = 3$ different mines and must deliver it to $N = 4$ different customers. Mines have a productive capacity Q_i $i = 1, \ldots, M$ and the coal is characterized by ash a_i and sulfur s_i content (per ton) and extraction cost c_i (euro / ton) that depend on the mine as reported in the table

	ash a_i ton	sulfur s_i ton	cost c_i euro/ton	capacity Q_i ton
mine 1	0,08	0,05	50	120
mine 2	0,06	0,04	55	100
mine 3	0,04	0,03	62	140

Customers have a coal request R_j $j = 1, \ldots, 4$ shown in the table

	Customer 1	Customer 2	Customer 3	Customer 4
request R_j	80	70	60	40

Shipping from a mine to a customer has a cost (euro / ton) c_{ij} reported in the table

	Customer 1	Customer 2	Customer 3	Customer 4
mine 1	4	6	8	12
mine 2	9	6	7	11
mine 3	8	12	3	5

Moreover, the coal mixture delivered to each individual customer j must have a maximum ash and sulfur contents respectively of 6 %, and 3.5 %.

Write an LP problem to find the optimal transportation strategy of the company.