OPERATIONS RESEARCH

EXAM November 8, 2017

IMPORTANT: READ CAREFULLY

The grade on the written exam is valid at most for three months.

Check the score of each exercise.

Report your answers to the exercises in the appropriate boxes below.

You must include the sheets with your calculus and explanations that will be used to verify the correct procedure.

Calculation errors that lead to a wrong answer may not be taken into account.

Please note that $\boxed{\text{YES}}$ $\boxed{\text{NO}}$ answers are NOT valid if you do not give an explanation.

Exercise 1. **(Score 7)** Consider the following nonlinear programming problem

$$\min_{x\in\mathbb{R}^3} \quad x_1^2 + 2x_2^2 + 3x_3^2 + 2x_1x_3 + 20x_1 + 2x_2 + 30x_3$$
$$\begin{aligned} & x_1 + 2x_2 + x_3 \leq 10 \\ & 2x_1 - x_2 + 3x_3 = 8 \\ & x_1, x_3 \geq 0 \end{aligned}$$

(i) **(score 1)** Consider the unconstrained problem (remove the constraints). Is there any unconstrained stationary point ? Report the points, if any and state whether they are a local minima/maxima or a saddle points.

(ii) **(score 1)** Are the KKT conditions necessary and sufficient for optimality ?

(iii) **(score 2)** Consider the point $\widehat{x} = (\frac{3}{2},\ -\frac{7}{2},\ \frac{1}{2})^T$ and write the KKT conditions in $\widehat{x}$. Are the KKT conditions satisfied ? Evaluate the multipliers

(iv) **(score 1)** Write the system to get a feasible and descent direction in $\widehat{x}$. Does a solution of the system (namely a feasible and descent direction) exist?

(v) **(score 2)** Find a a feasible direction along which it is possible to move from $\widehat{x} = (0,\ -\frac{7}{2},\ \frac{3}{2})^T$ finding an additional active constraint. Find the stepsize $t^{\max}$ and the corresponding new point y. Is the direction also a descent one?

Exercise 2. **(Score 3)** Consider the following Linear programming problem

$$\begin{aligned} \max \quad & -2x_1 + x_2 - 9x_3 \\ & -x_1 - 2x_2 + x_3 \geq -10 \\ & 2x_1 - x_2 - 3x_3 = 8 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

(i) (score 1) Check whether the following point are vertex of the feasible polyedron or not

$$x_A = \begin{pmatrix} 5 \\ 2 \\ 0 \end{pmatrix} \quad x_B = \begin{pmatrix} \frac{26}{5} \\ \frac{12}{5} \\ 0 \end{pmatrix} \quad x_C = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \quad x_D = \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix}$$

(ii) (score 0,5) Write the problem in the standard form for the simplex method. ($Ax = b,\ x \geq 0$ with $b \geq 0$).

(iii) (score 1,5) Write a BFS for the problems at point (ii) and specify the corresponding matrices B and N.

Exercise 3. (Score 7) Consider the following Linear programming problem

$$\begin{aligned} \max \quad & -2x_1 + x_2 - 9x_3 \\ & -x_1 - 2x_2 + x_3 \geq -10 \\ & 2x_1 - x_2 - 3x_3 = 8 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

(i) (score 2) Write the dual problem

(ii) (score 2) Solve graphically the dual: plot the feasible region, the level lines of the objective function, identify graphically the solution and find its value.

(iii) (score 2) Using duality theory, state if the primal problem has an optimal solution and in the affirmative case find it.

(iv) (score 1) State how the value of the optimal solution change if the rhs of the first constraint change from -10 to $-10 + \varepsilon$ with $\varepsilon > 0$ and sufficiently small.

Exercise 4. (score 5) Consider the following Multiobjective Linear Programming problem

$$\begin{aligned} \max \quad & \{-10x_1 + 8x_2, 2x_1 - 7x_2\} \\ & -x_1 + 2x_2 \leq 2 \\ & 2x_1 + x_2 \geq 1 \\ & x_1 - 3x_2 \leq 9 \\ & x_2 \leq 3 \\ & x_1 \geq 0 \end{aligned}$$

(i) (score 1) Find the two optimal point (x^{*1}, x^{*2}) of each of the two objectives and the corresponding ideal vector (z^{*1}, z^{*2})

(ii) (score 2) Plot the imagine of the feasible region into the Objective function space. Plot the Pareto front in the graph.

(iii) (score 2) Write the KKT conditions. Are the KKT satisfied in the point $\hat{x} = (1,\ \frac{1}{2})^T$? Justify your answer by evaluating the multipliers.

Exercise 5. (Score 4) Consider the following integer linear programming problem

$$
\begin{aligned}
\max \quad & 2x_1 - 7x_2 \\
& -x_1 + 2x_2 \leq 2 \\
& 2x_1 + x_2 \geq 1 \\
& x_1 - 3x_2 \leq 9 \\
& x_2 \leq 3 \\
& x_1 \geq 0 \\
& x \text{integer}
\end{aligned}
$$

Let $(1,\ -1)^T$ be a feasible point.

(i) (score 1) Solve the linear relaxation. Report the lower and the upper bounds to the optimal integer solution.

(ii) (score 1) Write the two subproblem $\mathcal{P}_1, \mathcal{P}_2$ obtained by branching with respect to fractional variable x_2

(iii) (score 1) Solve problem $\mathcal{P}_1$. Explain if it is possible to close it.

(iv) (score 1) Solve problem $\mathcal{P}_2$. Explain if it is possible to close it.

Exercise 6. (Score 5)

A company produces coal in $M = 3$ different mines and must deliver it to $N = 4$ different customers. Mines have a productive capacity Q_i $i = 1, \ldots, M$ and the coal is characterized by ash a_i and sulfur s_i content (per ton) and extraction cost c_i (euro / ton) that depend on the mine as reported in the table

	ash a_i ton	sulfur s_i ton	cost c_i euro/ton	capacity Q_i ton
mine 1	0,08	0,05	50	120
mine 2	0,06	0,04	55	100
mine 3	0,04	0,03	62	140

Customers have a coal request R_j $j = 1, \ldots, 4$ shown in the table

	Customer 1	Customer 2	Customer 3	Customer 4
request R_j	80	70	60	40

Shipping from a mine to a customer has a cost (euro / ton) c_{ij} reported in the table

	Customer 1	Customer 2	Customer 3	Customer 4
mine 1	4	6	8	12
mine 2	9	6	7	11
mine 3	8	12	3	5

Moreover, the coal mixture delivered to each individual customer j must have a maximum ash and sulfur contents respectively of 6 %, and 3.5 %.

Write an LP problem to find the optimal transportation strategy of the company.