

Knowledge representation and semantic technologies

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Exercises on Description Logics

Exercise 1 Given the following TBox:

$$\begin{aligned} A &\sqsubseteq B \\ B &\sqsubseteq C \\ C &\sqsubseteq \exists R.D \\ D &\sqsubseteq \neg A \end{aligned}$$

1. tell whether the TBox \mathcal{T} is satisfiable, and if so, show a model for \mathcal{T} ;
2. tell whether the concept D is satisfiable with respect to \mathcal{T} , and if so, show a model for \mathcal{T} where the interpretation of D is non-empty;
3. tell whether the concept expression $A \sqcap D$ is satisfiable with respect to \mathcal{T} , and if so, show a model for \mathcal{T} where the interpretation of $A \sqcap D$ is non-empty.

Solution

1. Let \mathcal{I} be the interpretation over the domain $\Delta^{\mathcal{I}} = \{d\}$ such that $A^{\mathcal{I}} = B^{\mathcal{I}} = C^{\mathcal{I}} = D^{\mathcal{I}} = R^{\mathcal{I}} = \emptyset$. It is immediate to see that all the axioms of \mathcal{T} are satisfied in \mathcal{I} : e.g., since $A^{\mathcal{I}}$ is empty, it is obviously true that $A^{\mathcal{I}} \subseteq B^{\mathcal{I}}$, hence the first axiom of \mathcal{T} is satisfied by \mathcal{I} . Consequently, \mathcal{I} is a model for \mathcal{T} , which implies that \mathcal{T} is satisfiable.
2. To prove that the concept D is satisfiable with respect to \mathcal{T} we have to show a model for \mathcal{T} where the interpretation of D is non-empty. Now, the above model \mathcal{I} does not show that D is satisfiable with respect to \mathcal{T} , because $D^{\mathcal{I}}$ is empty. So, we define a new interpretation \mathcal{J} , over the domain $\Delta^{\mathcal{J}} = \{d\}$, such that $A^{\mathcal{J}} = B^{\mathcal{J}} = C^{\mathcal{J}} = R^{\mathcal{J}} = \emptyset$ and $D^{\mathcal{J}} = \{d\}$. Again, it is immediate to verify that all the axioms of \mathcal{T} are satisfied in \mathcal{J} . In particular, $D \sqsubseteq \neg A$ is satisfied since $(\neg A)^{\mathcal{J}} = \Delta^{\mathcal{J}} = \{d\}$. Consequently, \mathcal{J} is a model for \mathcal{T} .
3. Since the TBox \mathcal{T} contains the axiom $D \sqsubseteq \neg A$, it follows that every model \mathcal{I} for \mathcal{T} is such that $D^{\mathcal{I}} \subseteq (\neg A)^{\mathcal{I}}$, i.e., $D^{\mathcal{I}} \cap A^{\mathcal{I}} = \emptyset$. Consequently, no model \mathcal{I} for \mathcal{T} exists such that $(A \sqcap D)^{\mathcal{I}}$ is non-empty.

Exercise 2 Given the knowledge base (KB) $\mathcal{K} = \langle \mathcal{T}, \mathcal{A} \rangle$, where \mathcal{T} is the following TBox:

$$\begin{aligned} (\text{Ax1}) \quad & A \sqsubseteq B \\ (\text{Ax2}) \quad & B \sqsubseteq C \\ (\text{Ax3}) \quad & C \sqsubseteq \exists R.D \\ (\text{Ax4}) \quad & D \sqsubseteq \neg A \\ (\text{Ax5}) \quad & A \sqsubseteq \forall R.A \end{aligned}$$

and \mathcal{A} is the following ABox:

$$\{ A(a), D(c), R(a, b), R(b, c) \}$$

1. using the tableau method, tell whether the KB \mathcal{K} is satisfiable (i.e., consistent), and if so, show a model for \mathcal{K} ;
2. now consider the KB \mathcal{K}' obtained from \mathcal{K} by deleting axiom (Ax1) in the TBox. Tell whether the concept assertion $\neg A \sqcup \neg D(c)$ is entailed by \mathcal{K}' , using the tableau method.

Solution, point 1

We start by considering point 1 of the exercise. First, \top_{GCI} for the given TBox is the following concept expression:

$$\top_{GCI} = (\neg A \sqcup B) \sqcap (\neg B \sqcup C) \sqcap (\neg C \sqcup \exists R.D) \sqcap (\neg D \sqcup \neg A) \sqcap (\neg A \sqcup \forall R.A)$$

Now, we start the tableau from the initial ABox $\mathcal{A}_0 = \mathcal{A}$:

$$\mathcal{A}_0 = \{ A(a), D(c), R(a, b), R(b, c) \}$$

We then apply the tableau \top_{GCI} -rule to the individual a , obtaining

$$\mathcal{A}_1 = \mathcal{A}_0 \cup \{ ((\neg A \sqcup B) \sqcap (\neg B \sqcup C) \sqcap (\neg C \sqcup \exists R.D) \sqcap (\neg D \sqcup \neg A) \sqcap (\neg A \sqcup \forall R.A))(a) \}$$

We then apply the tableau and-rule to the last assertion, obtaining

$$\mathcal{A}_2 = \mathcal{A}_1 \cup \{ (\neg A \sqcup B)(a), (\neg B \sqcup C)(a), (\neg C \sqcup \exists R.D)(a), (\neg D \sqcup \neg A)(a), (\neg A \sqcup \forall R.A)(a) \}$$

We then apply the tableau or-rule to the assertion $(\neg A \sqcup B)(a)$, obtaining

$$\begin{aligned} \mathcal{A}_3 &= \mathcal{A}_2 \cup \{ \neg A(a) \} \\ \mathcal{A}_4 &= \mathcal{A}_2 \cup \{ B(a) \} \end{aligned}$$

Now, \mathcal{A}_3 contains the clash $\{A(a), \neg A(a)\}$ (since $A(a) \in \mathcal{A}_0$), so it is closed. We then consider \mathcal{A}_4 and apply the tableau or-rule to the assertion $(\neg B \sqcup C)(a)$, obtaining

$$\begin{aligned} \mathcal{A}_5 &= \mathcal{A}_4 \cup \{ \neg B(a) \} \\ \mathcal{A}_6 &= \mathcal{A}_4 \cup \{ C(a) \} \end{aligned}$$

Now, \mathcal{A}_5 contains the clash $\{B(a), \neg B(a)\}$ (since $B(a) \in \mathcal{A}_4$), so it is closed. We then consider \mathcal{A}_6 and apply the tableau or-rule to the assertion $(\neg C \sqcup \exists R.D)(a)$, obtaining

$$\begin{aligned} \mathcal{A}_7 &= \mathcal{A}_6 \cup \{ \neg C(a) \} \\ \mathcal{A}_8 &= \mathcal{A}_6 \cup \{ \exists R.D(a) \} \end{aligned}$$

Now, \mathcal{A}_7 contains the clash $\{C(a), \neg C(a)\}$ (since $C(a) \in \mathcal{A}_6$), so it is closed. We then consider \mathcal{A}_8 and apply the tableau \exists -rule to the assertion $\exists R.D(a)$, obtaining

$$\mathcal{A}_9 = \mathcal{A}_8 \cup \{ R(a, x), D(x) \}$$

We now apply the tableau or-rule to the assertion $(\neg D \sqcup \neg A)(a)$, obtaining

$$\begin{aligned}\mathcal{A}_{10} &= \mathcal{A}_9 \cup \{ \neg D(a) \} \\ \mathcal{A}_{11} &= \mathcal{A}_9 \cup \{ \neg A(a) \}\end{aligned}$$

Now, \mathcal{A}_{11} contains the clash $\{A(a), \neg A(a)\}$ (since $A(a) \in \mathcal{A}_0$), so it is closed. We then consider \mathcal{A}_{10} and apply the tableau or-rule to the assertion $(\neg A \sqcup \forall R.A)(a)$, obtaining

$$\begin{aligned}\mathcal{A}_{12} &= \mathcal{A}_{10} \cup \{ \neg A(a) \} \\ \mathcal{A}_{13} &= \mathcal{A}_{10} \cup \{ \forall R.A(a) \}\end{aligned}$$

Again, \mathcal{A}_{12} contains the clash $\{A(a), \neg A(a)\}$ (since $A(a) \in \mathcal{A}_0$), so it is closed. We now consider \mathcal{A}_{13} and apply the tableau \forall -rule to the assertion $\forall R.A(a)$ (notice that $A(a)$ and $R(a, x)$ belong to \mathcal{A}_{13}), obtaining

$$\mathcal{A}_{14} = \mathcal{A}_{13} \cup \{ A(x) \}$$

We then apply the tableau \top_{GCI} -rule to the individual x , obtaining

$$\mathcal{A}_{15} = \mathcal{A}_{14} \cup \{ ((\neg A \sqcup B) \sqcap (\neg B \sqcup C) \sqcap (\neg C \sqcup \exists R.D) \sqcap (\neg D \sqcup \neg A) \sqcap (\neg A \sqcup \forall R.A))(x) \}$$

We then apply the tableau and-rule to the last assertion, obtaining

$$\mathcal{A}_{16} = \mathcal{A}_{15} \cup \{ (\neg A \sqcup B)(x), (\neg B \sqcup C)(x), (\neg C \sqcup \exists R.D)(x), (\neg D \sqcup \neg A)(x), (\neg A \sqcup \forall R.A)(x) \}$$

We then apply the tableau or-rule to the assertion $(\neg D \sqcup A)(x)$, obtaining

$$\begin{aligned}\mathcal{A}_{17} &= \mathcal{A}_{16} \cup \{ \neg D(x) \} \\ \mathcal{A}_{18} &= \mathcal{A}_{16} \cup \{ \neg A(x) \}\end{aligned}$$

Now, notice that $D(x) \in \mathcal{A}_9$, therefore \mathcal{A}_{17} contains the clash $D(x), \neg D(x)$. Moreover, notice that $A(x) \in \mathcal{A}_{14}$, therefore \mathcal{A}_{18} contains the clash $A(x), \neg A(x)$.

Consequently, all the ABoxes (branches) generated by the tableau are closed. We can thus conclude that the knowledge base $\mathcal{K} = \langle \mathcal{T}, \mathcal{A} \rangle$ of point 1 is inconsistent (unsatisfiable).

Solution, point 2

We now consider point 2 of the exercise. First, \top_{GCI} for the given TBox is the following concept expression:

$$\top_{GCI} = (\neg B \sqcup C) \sqcap (\neg C \sqcup \exists R.D) \sqcap (\neg D \sqcup \neg A) \sqcap (\neg A \sqcup \forall R.A)$$

Now, we start the tableau from the initial ABox \mathcal{A}_0 obtained by the ABox \mathcal{A} of point 1 adding the negation of the assertion $\neg A \sqcup \neg D(c)$:

$$\mathcal{A}_0 = \mathcal{A} \cup \{ A \sqcap \neg D(c) \}$$

We apply the tableau and-rule to the above assertion, obtaining

$$\mathcal{A}_1 = \mathcal{A}_0 \cup \{ A(c), D(c) \}$$

We then apply the tableau \top_{GCI} -rule to the individual c , obtaining

$$\mathcal{A}_2 = \mathcal{A}_1 \cup \{ (\neg B \sqcup C) \sqcap (\neg C \sqcup \exists R.D) \sqcap (\neg D \sqcup \neg A) \sqcap (\neg A \sqcup \forall R.A)(c) \}$$

We then apply the tableau and-rule to the last assertion, obtaining

$$\mathcal{A}_3 = \mathcal{A}_2 \cup \{ (\neg B \sqcup C)(a), (\neg C \sqcup \exists R.D)(a), (\neg D \sqcup \neg A)(a), (\neg A \sqcup \forall R.A)(c) \}$$

We then apply the tableau or-rule to the assertion $(\neg D \sqcup \neg A)(c)$, obtaining

$$\begin{aligned} \mathcal{A}_4 &= \mathcal{A}_3 \cup \{ \neg D(c) \} \\ \mathcal{A}_5 &= \mathcal{A}_3 \cup \{ \neg A(c) \} \end{aligned}$$

Now, \mathcal{A}_4 contains the clash $\{D(c), \neg D(c)\}$ (since $D(c) \in \mathcal{A}_1$), so it is closed. Moreover, \mathcal{A}_5 contains the clash $\{A(c), \neg A(c)\}$ (since $A(c) \in \mathcal{A}_1$), so it is closed too.

Consequently, all the ABoxes (branches) generated by the tableau are closed. We can thus conclude that the knowledge base \mathcal{K}' entails the assertion $\neg A \sqcup \neg D(c)$.

Exercise 3 Given the knowledge base (KB) $\mathcal{K} = \langle \mathcal{T}, \mathcal{A} \rangle$, where \mathcal{T} is the following TBox:

$$\begin{aligned} A &\sqsubseteq B \sqcup C \\ B &\sqsubseteq \exists R.D \\ C &\sqsubseteq \exists R.E \\ A &\sqsubseteq \forall R.F \\ D \sqcap F &\sqsubseteq G \end{aligned}$$

and \mathcal{A} is the following ABox:

$$A(a)$$

1. using the tableau method, tell whether the concept assertion $\exists R.(E \sqcap G)(a)$ is entailed by \mathcal{K} ;
2. using the tableau method, tell whether the concept assertion $(\exists R.E) \sqcap (\exists R.G)(a)$ is entailed by \mathcal{K} .