Ottimizzazione Combinatoria 2

Cammino Minimo Vincolato

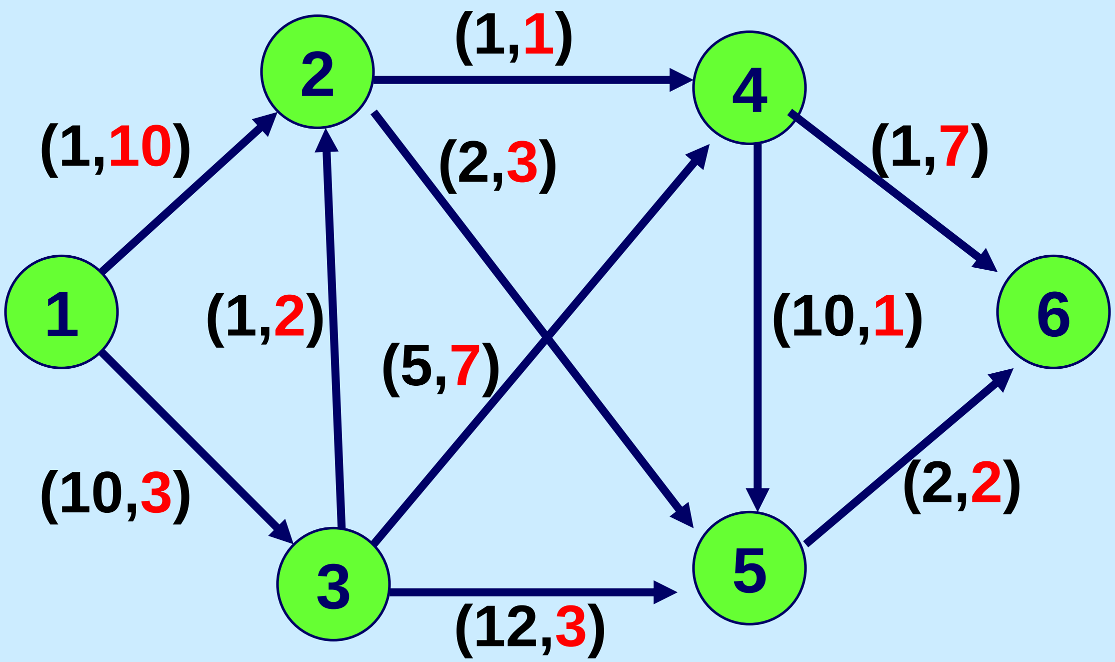
Un esempio basato sulle slide della seguente lezione del Corso Network Optimization MITOpenCourseware
http://ocw.mit.edu/courses/sloan-school-of-management/15-082j-network-optimization-fall-2010/

15.082J and 6.855J and ESD.78J

Lagrangian Relaxation 2

- **Applications**
- **Algorithms**
- **Theory**

The Constrained Shortest Path Problem



Find the *path of minimum cost* (*black*) from node 1 to node 6 with a *transit time (red)* at most 14.

Constrained Shortest Paths: Path Formulation

Given: a network $G = (N,A)$

w_{ij}	cost for arc (i,j)
$w(P)$	cost of path P
t_{ij}	traversal time for arc (i,j)
T	upper bound on transit times.
$t(P)$	traversal time for path P
$\mathbf{P}$	set of paths from node 1 to node n
	(*nodi 1 e 6 nell'esempio*)

$$\begin{aligned} \text{Min} \quad & w(P) \\ \text{s.t.} \quad & t(P) \le T \\ & P \in \mathbf{P} \end{aligned}$$

Constrained Problem

$$\begin{aligned} L(\mu) = \text{Min} \quad & w(P) + u\,(t(P) - T) \\ \text{s.t.} \quad & P \in \mathbf{P} \end{aligned}$$

Lagrangian

The Lagrangian Multiplier Problem

Step 0. Formulate the Lagrangian Problem.

$$L(u) = \text{Min} \quad w(P) + u\, t(P) - uT$$
$$\text{s.t.} \quad P \in \mathbf{P}$$

Step 1. Rewrite as a maximization problem

$$L(u) = v^* = \text{Max } v$$
$$\text{s.t } \quad v \leq w(P) + u\, t(P) - uT \quad \text{for all } P \in \mathbf{P}$$

Step 2. Write the Lagrangian multiplier problem

$$L^* = \max \{L(u) : u \geq 0\} =$$
$$= \quad \text{Max } v$$
$$\text{s.t } \quad v \leq w(P) + u\, t(P) - u\, T \quad \text{for all } P \in \mathbf{P}$$
$$u \geq 0$$

$$\text{Max}\ \{ v:\ v \le w(P) + u\,t(P) - uT \quad \forall P \in \mathbf{P},\ u \ge 0\}$$

$$-uT + \text{Min}\ \{ w(P) + u\,t(P)\ :\ P \in \mathbf{P} \} \rightarrow P^* = \text{Path}(u)$$

Rappresentiamo un cammino P con il suo ***vettore di incidenza*** x ($x_{ij} = 1 \leftrightarrow ij \in P$) e sia Q_{st} il *poliedro dei cammini minimi* tra I nodi ***s*** e ***t*** (nell'esempio 1 e 6).

Il problema corrispondente ad un ***fissato u*** si scrive:

$$-uT + min\left\{\sum_{ij\in A} w_{ij}x_{ij} + u\sum_{ij\in A} t_{ij}x_{ij} : \quad x \in Q_{st} \cap \{0,1\}^{|A|}\right\}$$

Ovvero:

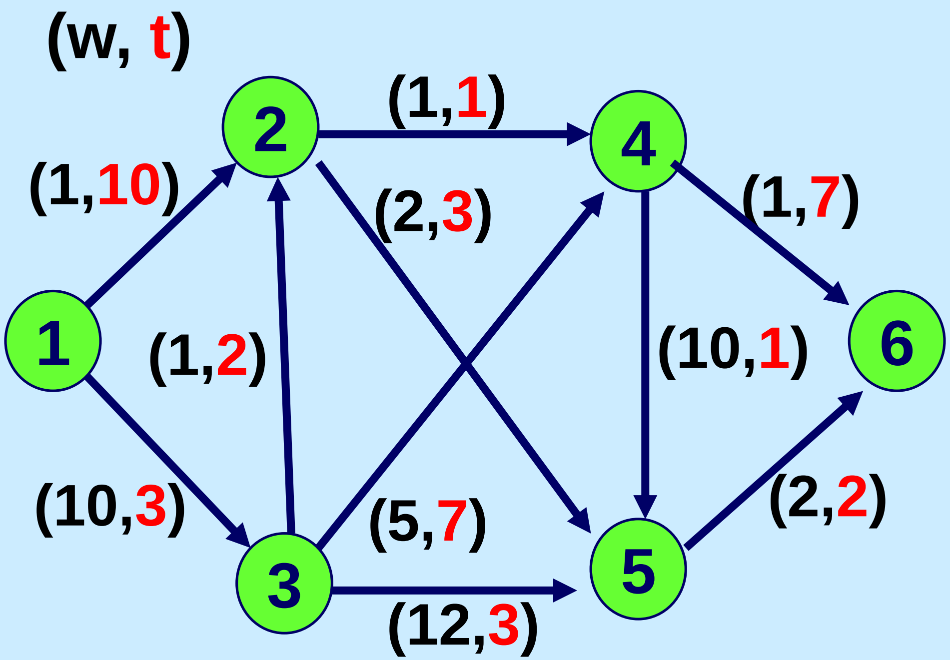
$$L(u) = -uT + min\left\{\sum_{ij\in A} (w_{ij}+ut_{ij})x_{ij} : \quad x \in Q_{st} \cap \{0,1\}^{|A|}\right\}$$

Per u fissato il problema lagrangiano è un problema di cammino minimo con pesi $(w_{ij}+ut_{ij})$

$\text{Max} \{ v: \; v \le w(P) + u\, t(P) - uT \quad \forall P \in \mathbf{P}, \; u \ge 0 \}$

$-uT + \text{Min} \{ w(P) + u\, t(P) \; : \; P \in \mathbf{P} \} \rightarrow P^* = \text{Path}(u)$

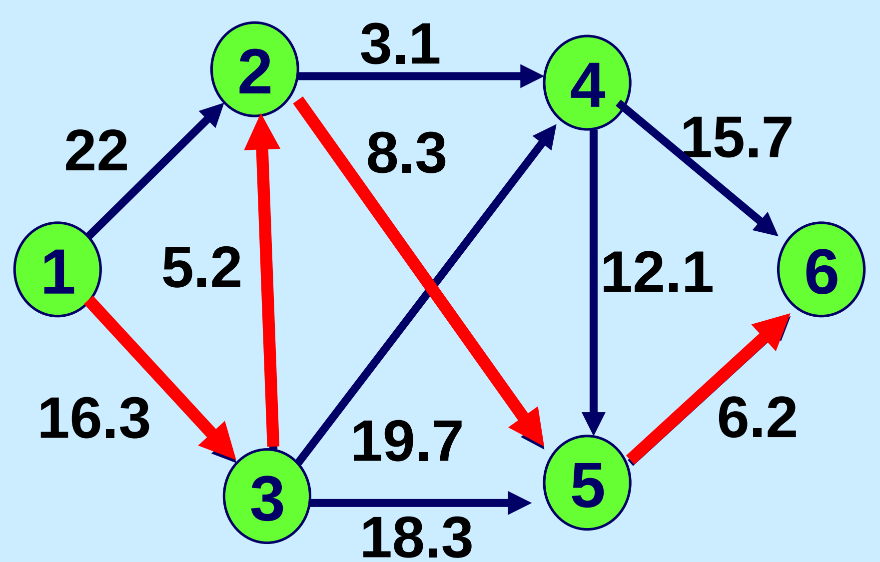
Per u fissato → Cammino minimo con pesi $(w_{ij} + u t_{ij})$



$u = 0 \rightarrow w_{ij}$

$u = M >> 0 \rightarrow w_{ij} + M\, t_{ij} \approx M\, t_{ij}$

$u = 2.1$



Per ogni ***u*** una **diversa istanza**
del cammino minimo

Discretizzazione del problema lagrangiano

$$Q \cap \{0,1\}^n = \{x^1, x^2, \ldots, x^p\} \quad \Rightarrow \quad x^k \in Q_{st} \cap \{0,1\}^{|A|}$$

x^k Vettore di incidenza di un cammino s-t (1-n)

$$L(u) = \min_{k=1,\ldots,p} \left\{ w^T x^k + u^T (b - Ax^k) \right\} = \min_{k=1,\ldots,p} \left\{ L^k(u) \right\}$$

$$L^k(u) = w^T x^k + u^T (b - Ax^k) \quad \Rightarrow \quad w(P) + u\,(t(P) - T)$$

Iperpiano in R^{m+1}

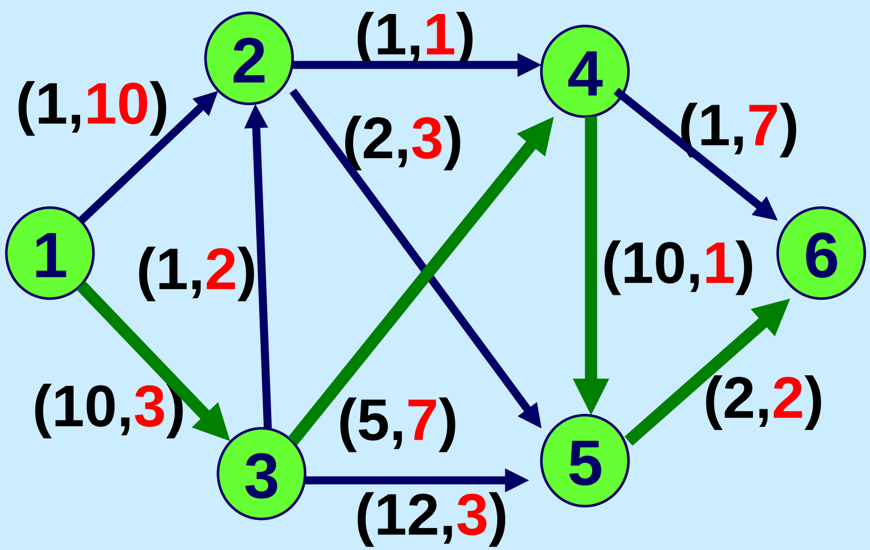
$$L^k(u) = \sum_{ij \in A} w_{ij} x_{ij}^k + u \left(\sum_{ij \in A} t_{ij} x_{ij}^k - T \right);$$

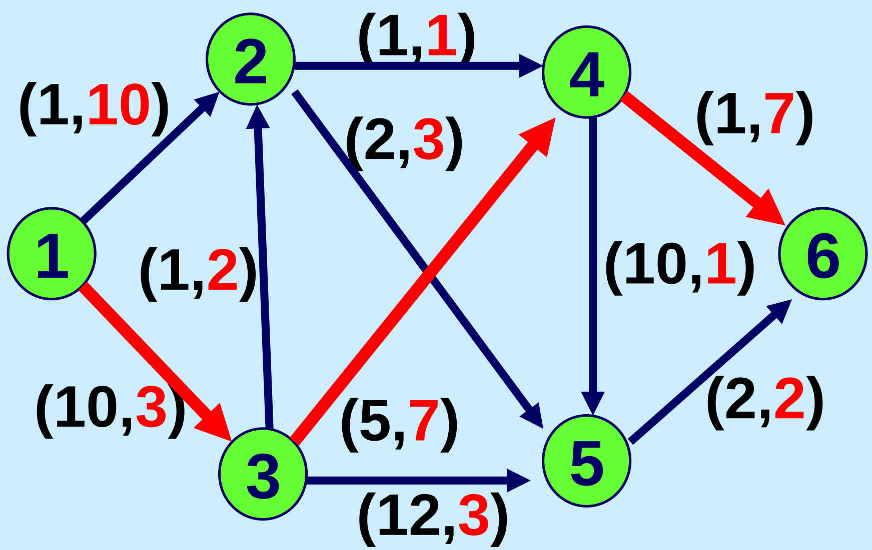
Positivo se il vincolo è violato

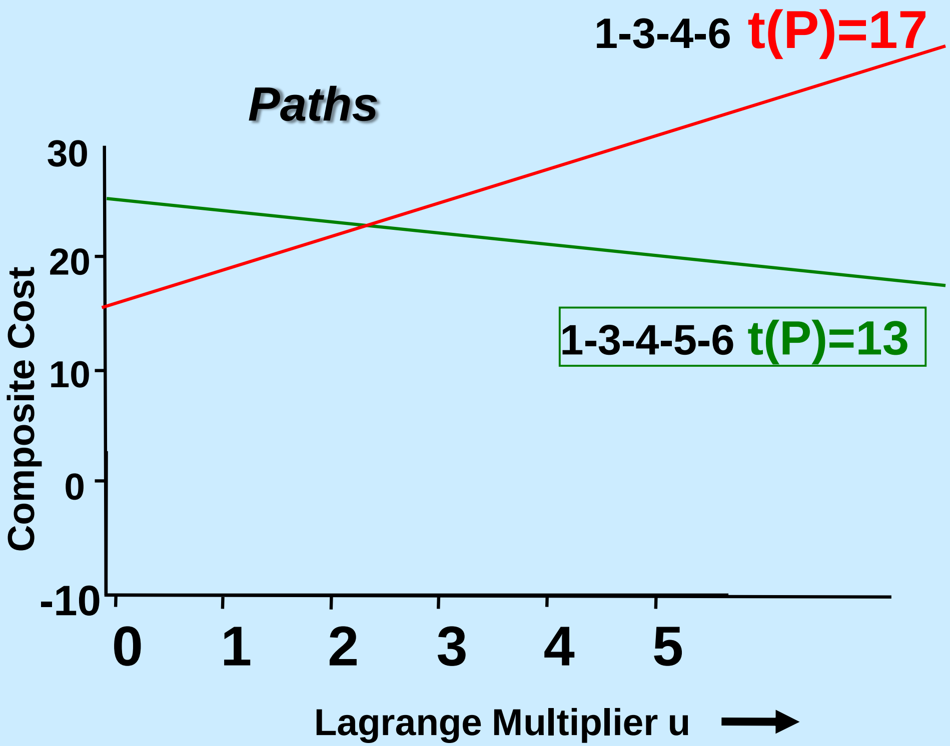
Ogni soluzione x^k (cammino) definisce un iperpiano (retta)

Discretizzazione del problema lagrangiano

Ogni soluzione (cammino) definisce un iperpiano (retta)







w(P) + u (t(P) – T)

27 + u (13 – 14) [pendenza neg]

16 + u (17 – 14) [pendenza pos]

$$\text{Max} \{ v: \ v \le w(P) + u\,t(P) - uT \quad \forall P \in \mathbf{P}, \text{ and } u \ge 0\}$$

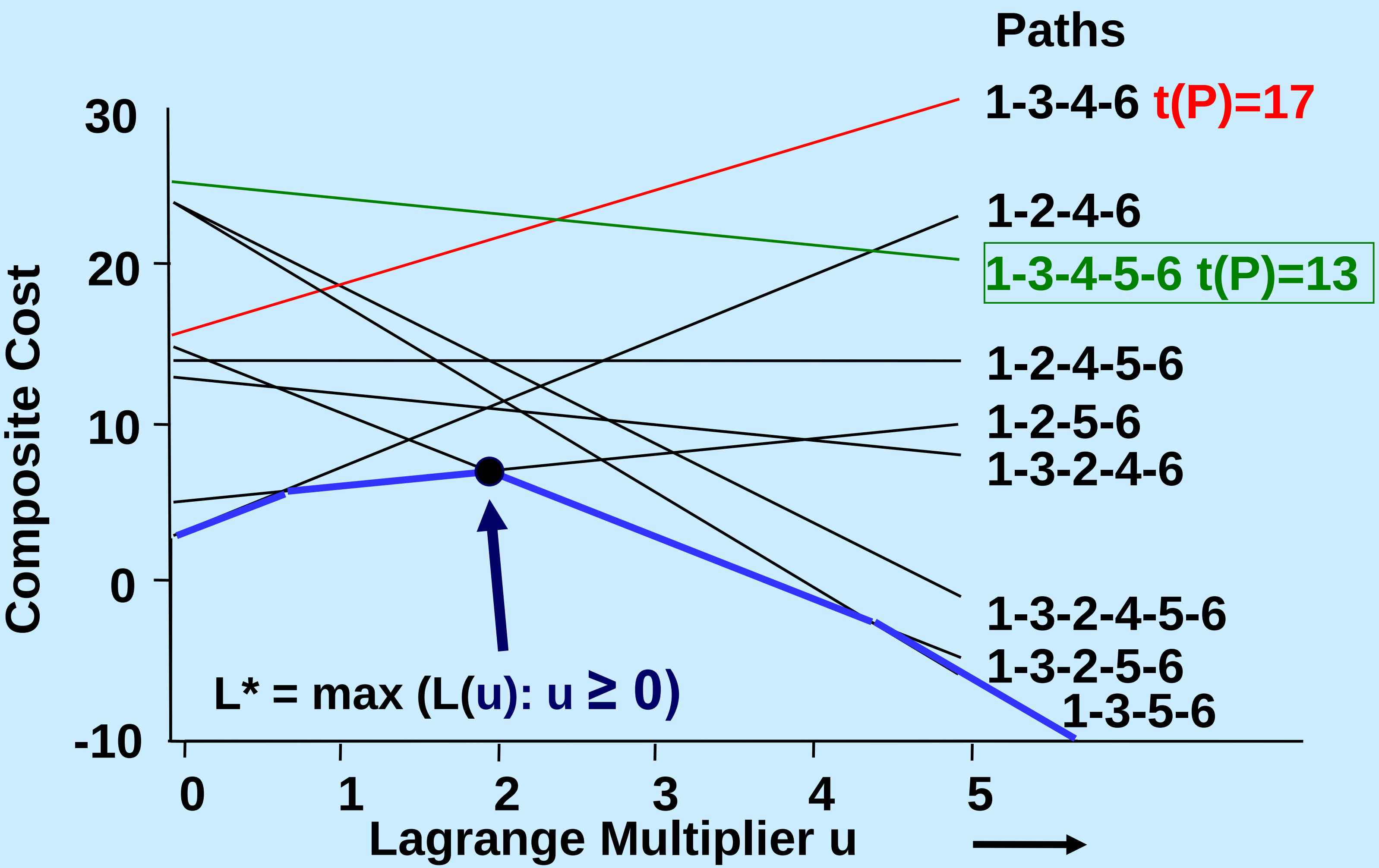


Figure 16.3 The Lagrangian function for T = 14.

The Restricted Lagrangian

$\mathcal{P}$: the set of paths from node 1 to node n

$\mathcal{B} \subseteq \mathcal{P}$: a subset of paths

$L^* = v^* = \max v$

s.t $v \leq w(P) + u\,(\,t(P) - T\,)$

for all $P \in \mathcal{P}$

$u \geq 0$

Lagrangian Multiplier Problem

$L^*_B = \max v$

s.t $v \leq w(P) + u\,t(P) - u\,T$

for all $P \in \mathcal{B}$

$u \geq 0$

Restricted Lagrangian Multiplier Problem

$L(u) \leq L^* \leq L^*_B$

If $L(u) = L^*_B$ then $L(u) = L^*$.

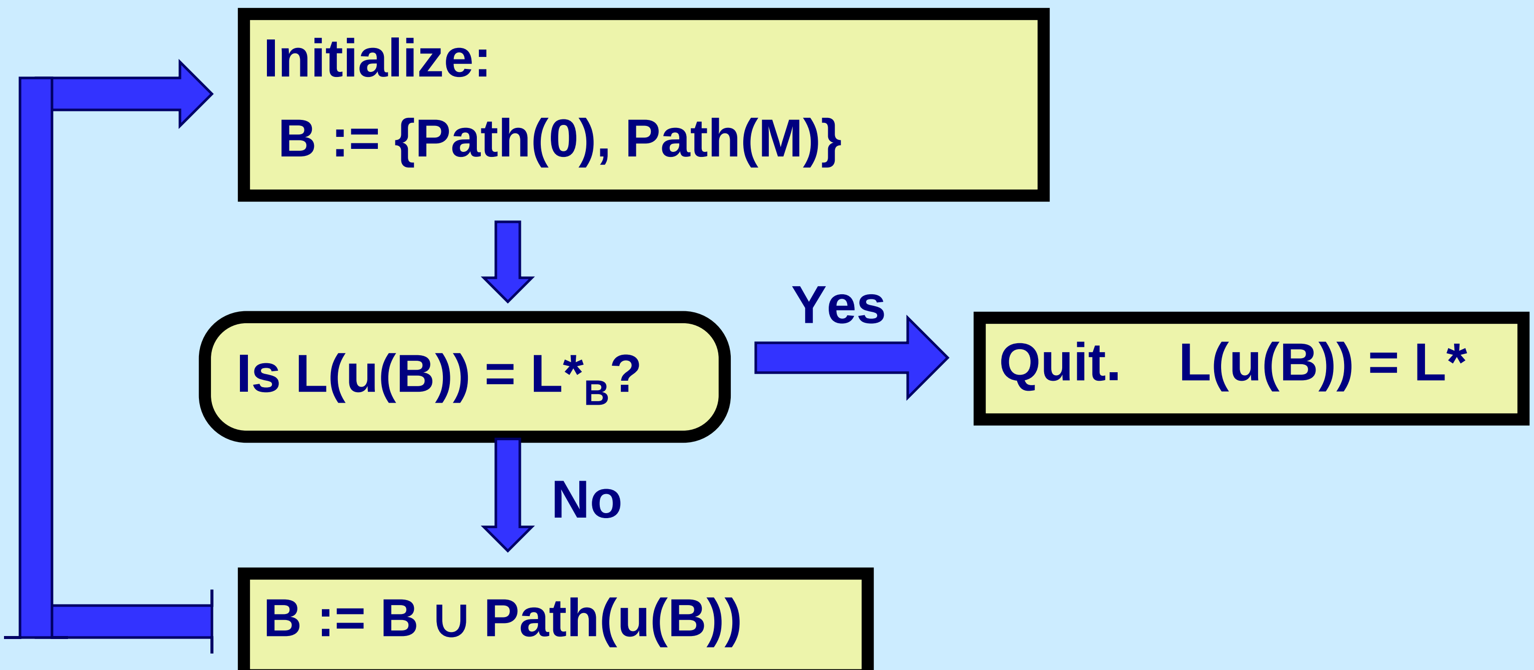
Optimality Conditions

Constraint Generation for Finding L*

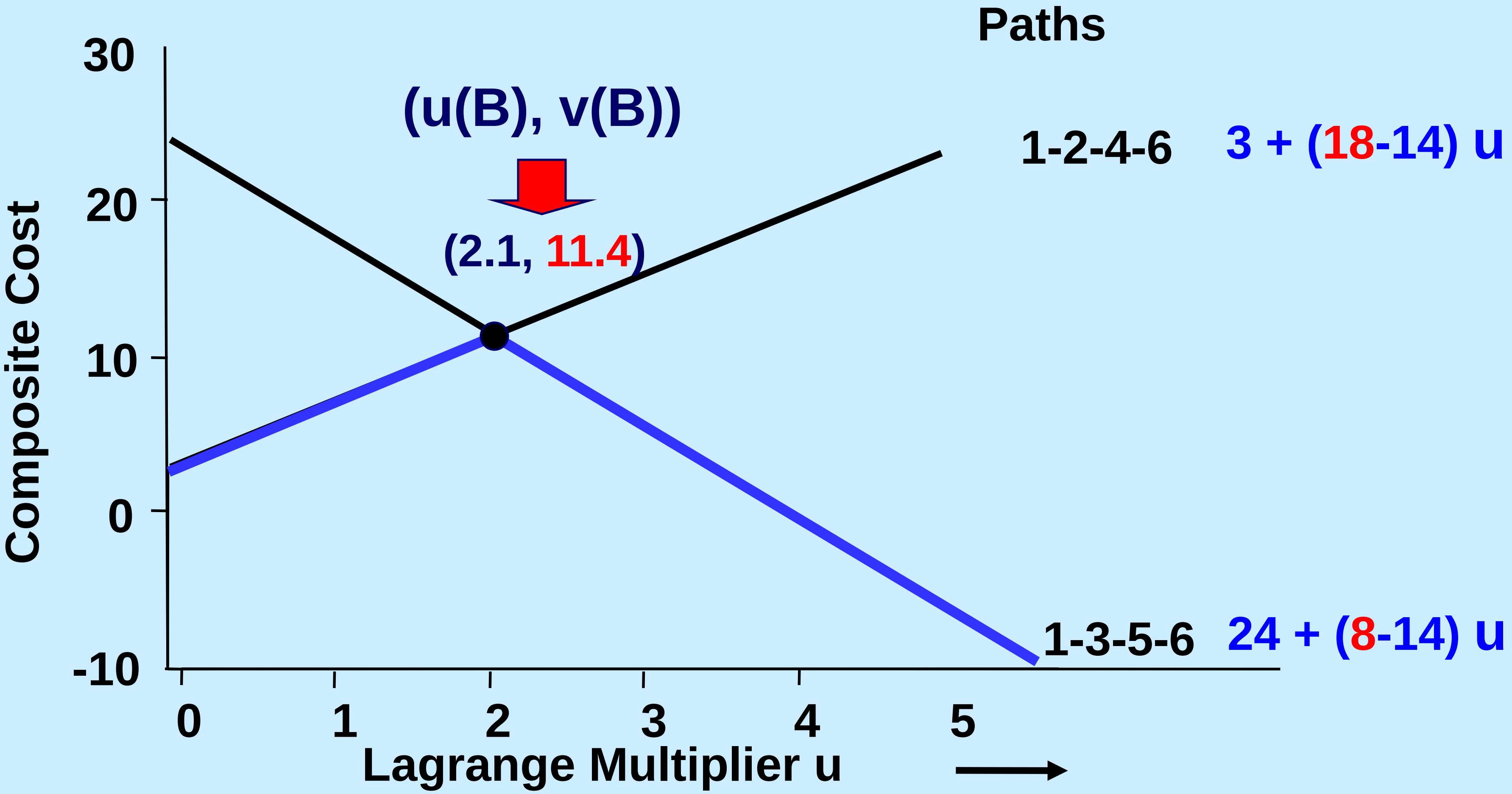
Let Path(u) be the path (*iperpiano*) that optimizes L(u).

Let u(B) be the value of u that optimizes $L_B(u)$.

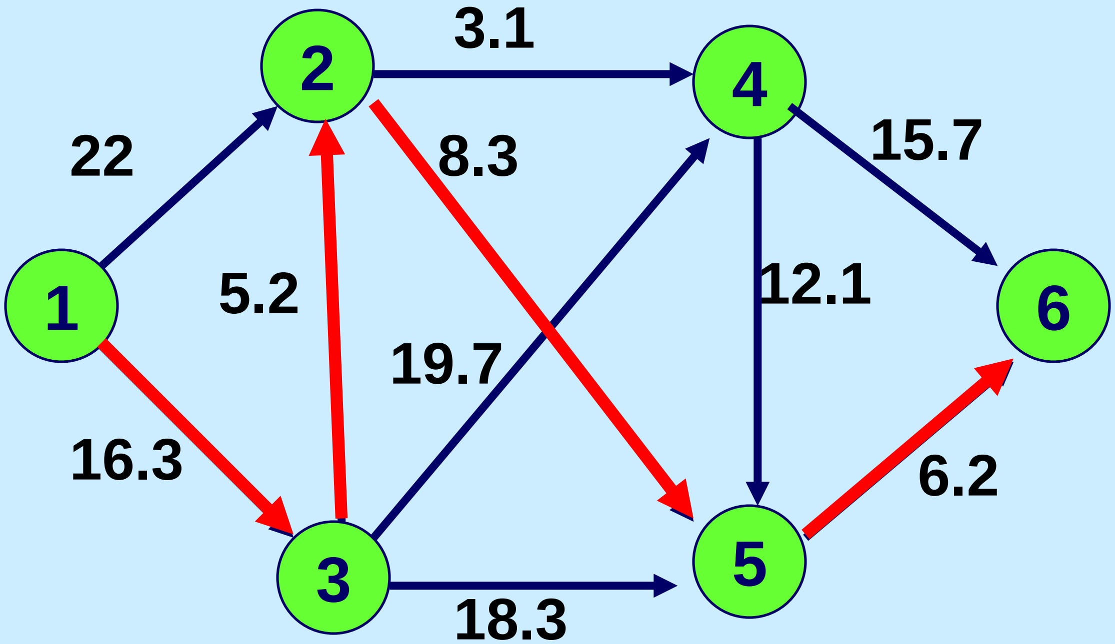
M is some large number



We start with the paths 1-2-4-6, and 1-3-5-6 which are optimal for L(0) and L(∞).

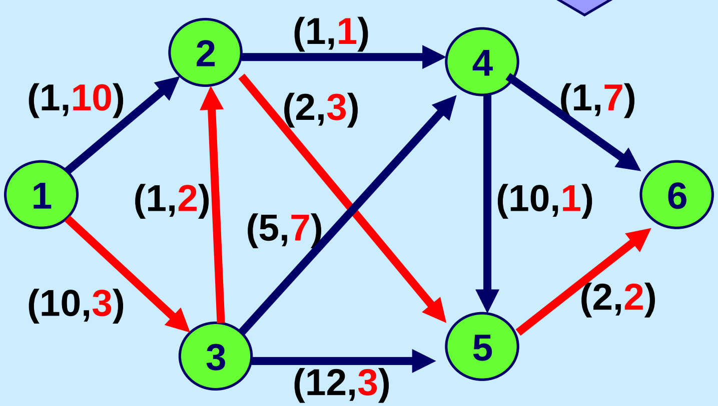


Set u(B) = 2.1 and solve the CSP problem *(L(u(B)))*



The optimum path P^* is 1-3-2-5-6

Costo modificato = 36
Costo = 15
T. Transito = 10



$$L\big(u(B)\big) = -u(B)T + w(P^*) + u(B)t(P^*) = \\ = -2.1 * 14 + 36 = -29.4 + 36 = 6.6$$

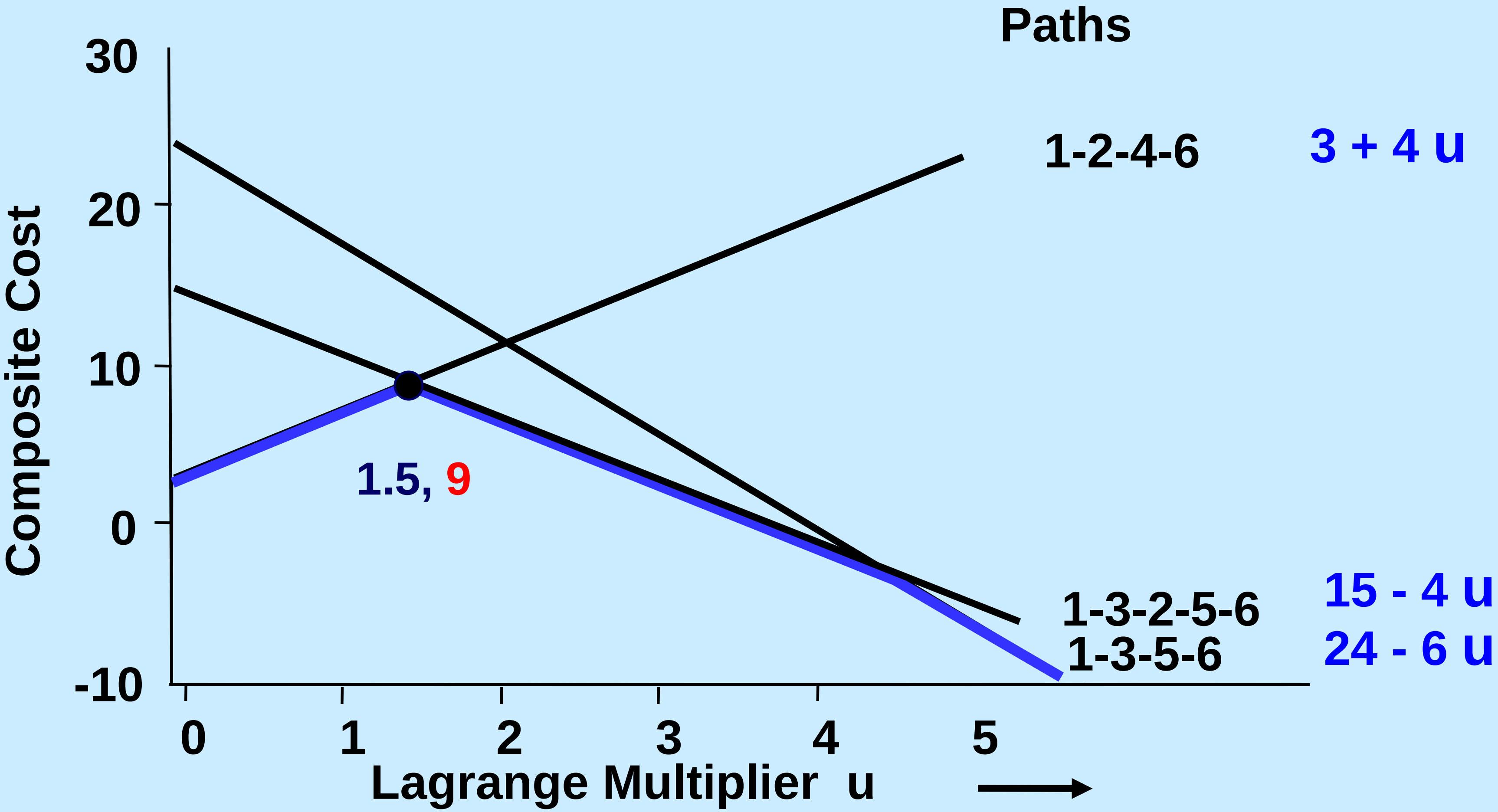
Ovvero:

L(u(B)) = 15 + 10 u(B) – 14 u(B)

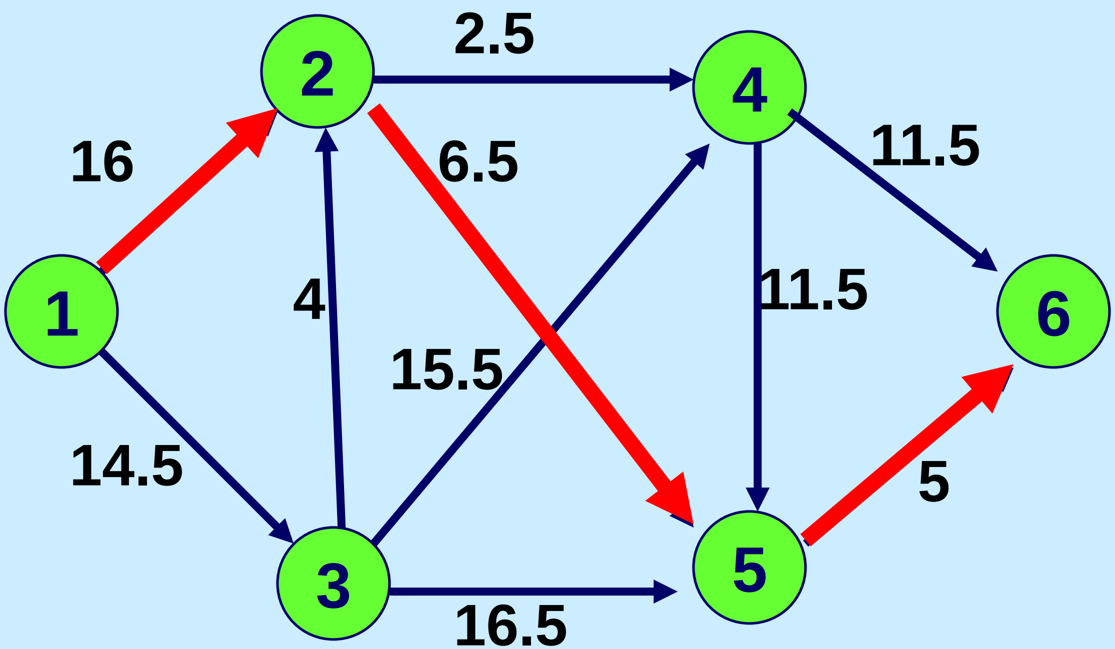
15 + 21 – 29.4 = 6.6

6.6 < 11.4 ⇒ *Aggiungi il path P^* e riottimizza*

Path(2.1) = 1-3-2-5-6. Add it to S and reoptimize.

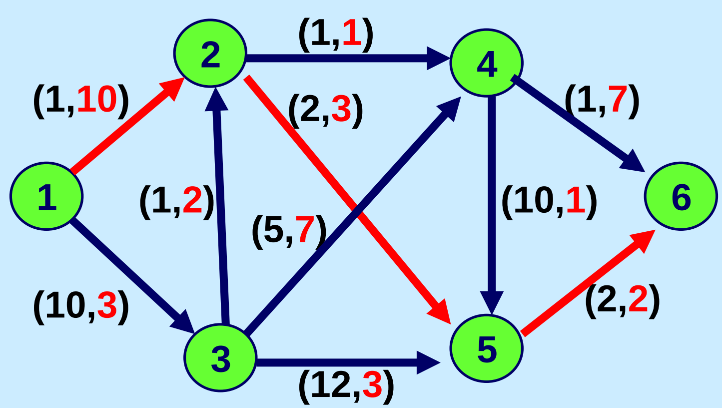


Set u(B) = 1.5 and solve the CSP problem *(L(u(B))*



The optimum path P^* is 1-2-5-6.

Costo modificato = 27.5
Costo = 5
T. Transito = 15

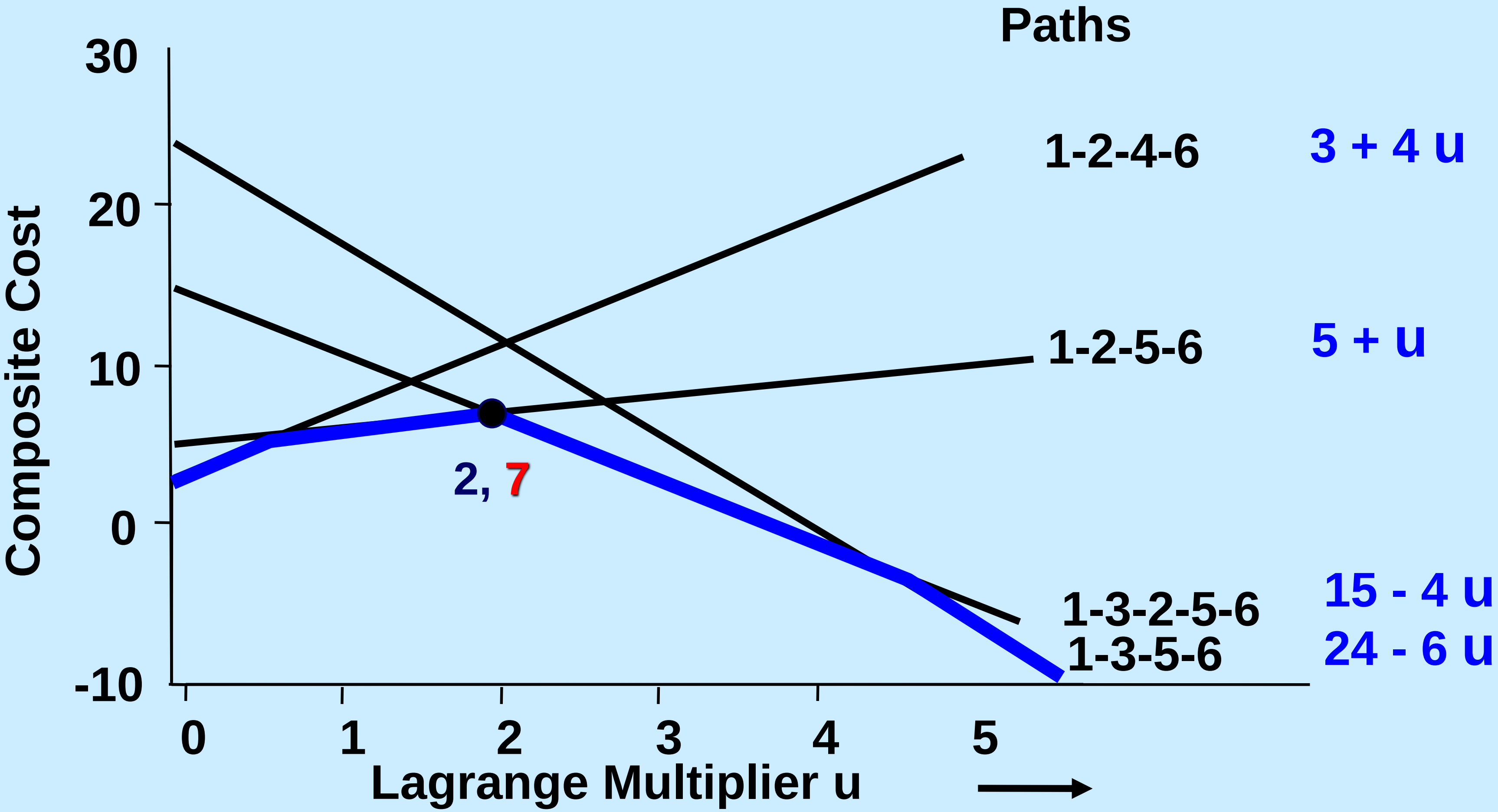


L(u(B)) = 5 + 15 u(B) – 14 u(B)

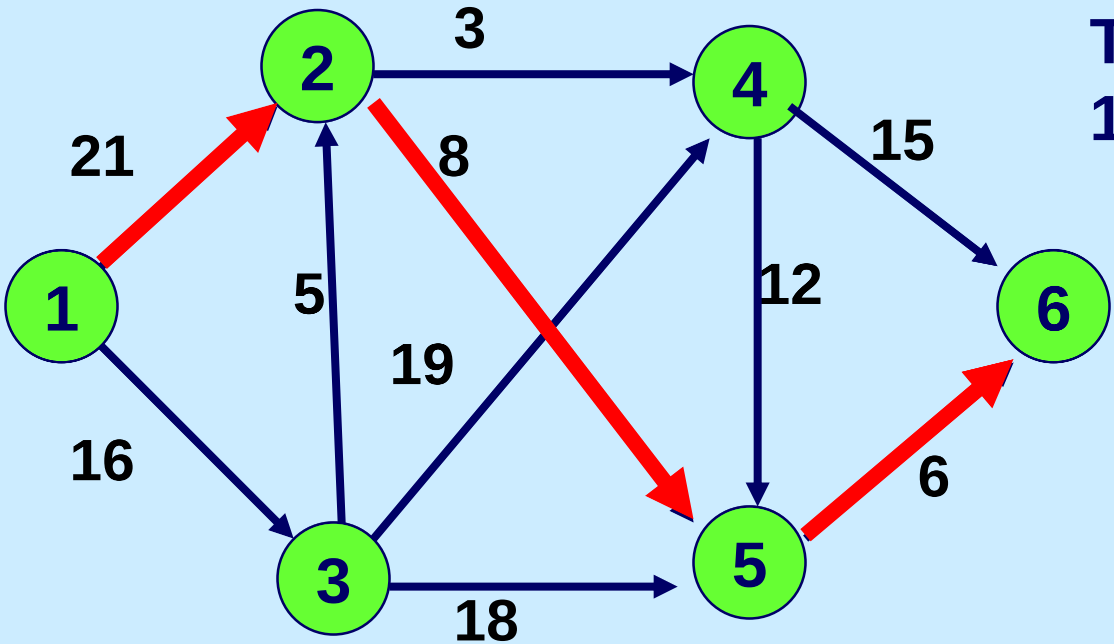
5 + 22.5 – 21 = 6.5

6.5 < 9 → *Aggiungi il path P^* e riottimizza*

Add Path 1-2-5-6 and reoptimize



Set u(B) = 2 and solve the CSP problem *(L(u(B))*



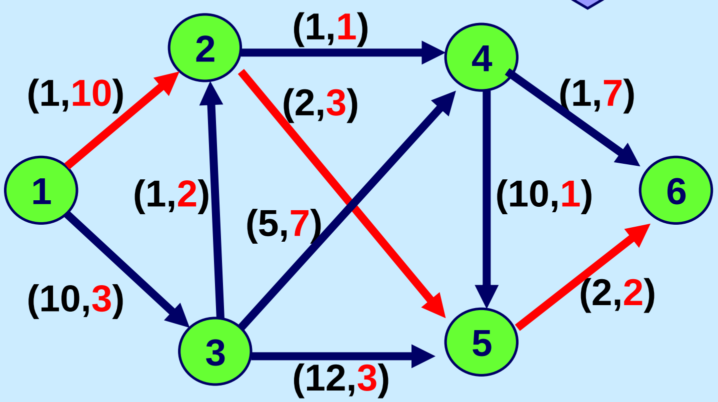
The optimum paths are 1-2-5-6 and 1-3-2-5-6

Scegliamo 1-2-5-6

Costo modificato = 35

Costo = 5

T. Transito = 15



L(u(B)) = 5 + 15 u(B) – 14 u(B)

= 5 + 30 – 28 = 7

7 = v(B) = 7

Ottimo del Duale Lagrangiano

There are no new paths to add.
u(B) = u* is optimal for the multiplier problem