A sound and complete tableau calculus for reasoning about only knowing and knowing at most

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Abstract. We define a tableau calculus for the logic of only knowing and knowing at most $\mathcal{ONL}$, which is an extension of Levesque's logic of only knowing $\mathcal{OL}$. The method is based on the possible-world semantics of the logic $\mathcal{ONL}$, and can be considered as an extension of known tableau calculi for modal logic K45. From the technical viewpoint, the main features of such an extension are the explicit representation of "unreachable" worlds in the tableau, and an additional branch closure condition implementing the property that each world must be either reachable or unreachable. The calculus allows for establishing the computational complexity of reasoning about only knowing and knowing at most. Moreover, we prove that the method matches the worst-case complexity lower bound of the satisfiability problem for both $\mathcal{ONL}$ and $\mathcal{OL}$. With respect to [22], in which the tableau calculus was originally presented, in this paper we both provide a formal proof of soundness and completeness of the calculus, and prove the complexity results for the logic $\mathcal{ONL}$.

1. Introduction

Epistemic logics for commonsense reasoning are generally based on the idea of providing systems (agents) with the ability of introspecting on their own knowledge and ignorance [17, 13]. To this aim, an *epistemic closure assumption* is generally adopted, which informally states that the logical theory formalizing the agent is a *complete* specification of the agent's knowledge. As a consequence, any fact that is not entailed by such a theory, according to a given semantics, is assumed to be *not known* by the agent. Roughly speaking, there exist two different ways to modify a monotonic logical framework in order to embed such a principle: either by modifying the semantics of the logic, thus realizing at the meta-level such a knowledge closure, or by representing the closure assumption *explicitly* in the language of the logic, suitably extending its syntax and semantics. The first approach has been pursued in the definition of several *modal* formalizations of nonmonotonic reasoning, e.g. McDermott and Doyle's nonmonotonic modal logics [15], Halpern and Moses' logic of minimal epistemic states [10] and Lifschitz's logic of minimal belief and negation as failure [14]. On the other hand, the second approach has been followed by Levesque [13] in the definition

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of the logic of *only knowing* ($\mathcal{OL}$), obtained by adding a second modal operator O interpreted in terms of “all-I-know”, to modal logic K45. Informally, such an interpretation of the modality O is obtained by maximizing the set of successors of each world satisfying O-formulae.

In a nutshell, the logic of only knowing is a monotonic formalism, in which the modality O allows for an explicit representation of the epistemic closure assumption at the object level (i.e. in the language of the logic), whereas in nonmonotonic formalisms the closure assumption is a meta-level notion. E.g., let φ be a modal formula specifying the knowledge of the agent: in the logic of only knowing, satisfiability of the formula $O\varphi$ in a world w requires maximization of the possible worlds connected to w and satisfying φ; an analogous kind of maximization is generally realized by the preference semantics of nonmonotonic modal logics, by choosing, among the models for φ, only the models having a “maximal” set of possible worlds, where such a notion of maximality changes according to the different proposals.

While the problem of finding a complete axiomatization for the logic $\mathcal{OL}$ has been extensively analyzed [13, 8, 9], few studies have analyzed the computational aspects of (and/or reasoning methods for) reasoning about only knowing. Indeed, the computational complexity of reasoning about only knowing in the propositional case has been only recently established [21].

In this paper we present a tableau method for the modal propositional fragment of Levesque’s logic of only knowing $\mathcal{OL}$. More precisely, we define a tableau calculus for the logic of only knowing and knowing at most $\mathcal{ONL}$ [13, 9], which extends $\mathcal{OL}$ with a third modality N interpreted in terms of “knowing at most”. Informally, the meaning of a formula $N\varphi$ in $\mathcal{ONL}$ is “I know at most $\neg\varphi$”, which is realized, in terms of a Kripke-style semantics, by imposing that $N\varphi$ is satisfied in a world w if and only if all worlds satisfying $\neg\varphi$ are connected to w.

The method is strictly based on the possible-world semantics of the logic $\mathcal{ONL}$, and can be considered as an extension of known tableau calculi for the modal logic K45. From the technical viewpoint, the main feature of such an extension is the explicit representation of “unreachable” worlds in the tableau, which allows for a proper handling of N-subformulae in the tableau. However, the explicit representation of unreachable worlds requires an additional branch closure condition in the calculus, which implements the restriction that *each* possible world must be either reachable or unreachable from the initial world. Such a condition is decided by means of a second-level tableau, which looks for the existence of a world that can be neither reachable nor unreachable from the initial world of any model consistent with the branch of the main tableau. Hence, if such a world exists (i.e., the second-level tableau is open), then the branch of the main tableau is closed.

Our tableau calculus allows for establishing the computational complexity of reasoning about only knowing and knowing at most: in particular, we prove that satisfiability in the modal propositional fragment of $\mathcal{ONL}$ is Σ_2^p-complete, while validity is Π_2^p-complete. We also prove that our method matches the worst-case complexity lower bound of the satisfiability problem for both $\mathcal{ONL}$ and $\mathcal{OL}$, and in this sense it can be considered "optimal" for these logics.

We remark that, due to its powerful expressive capabilities, the logic of only knowing is generally considered as a very general formalism for nonmonotonic reasoning. In particular, it has been proven [3] that such a logic is able to naturally embed several well-known nonmonotonic formalisms, i.e., autoepistemic logic, prerequisite-free default logic, disjunctive logic programming under stable model semantics, and circumscription. Therefore, our method can be also seen as a general, semantic-based calculus for nonmonotonic reasoning.

In the following, we first briefly introduce the modal logic of only knowing and knowing at most $\mathcal{ONL}$. Then, in Section 3 we present the tableau calculus for $\mathcal{ONL}$, and in Section 4 we prove correctness of the method. In Section 5 we analyze the computational properties of our method, and establish the complexity of reasoning in $\mathcal{ONL}$. Finally, we discuss related work and draw some conclusions in Section 6. With respect to [22], in which the tableau calculus was originally presented, in this paper we both provide a formal proof of soundness and completeness of the calculus, and prove the complexity results for the logic $\mathcal{ONL}$.

2. The logic $\mathcal{ONL}$

In this section we briefly recall the formalization of only knowing [13]. We assume that the reader is familiar with the basic notions of modal logic. We recall that K45 denotes the modal logic interpreted on Kripke structures whose accessibility relation among worlds is transitive and Euclidean (see e.g. [11] for more details).

We use $\mathcal{L}$ to denote a fixed propositional language with propositional connectives $\wedge, \neg$ (the symbols $\vee, \supset, \equiv$ are used as abbreviations), and whose generic atoms are elements of a countably infinite alphabet $\mathcal{A}$ of propositional symbols. We assume that $\mathcal{A}$ contains the symbols true, false. An interpretation (also called *world*) over $\mathcal{L}$ is a function that assigns a truth value to every atom of $\mathcal{L}$. For each interpretation w, $w(\mathsf{true}) = \mathit{TRUE}$ and $w(\mathsf{false}) = \mathit{FALSE}$. The interpretation of a propositional formula in an interpretation is defined in the usual way. We say that a formula $\varphi \in \mathcal{L}$ is *satisfiable* if there exists an interpretation w such that $w(\varphi) = TRUE$ (which we also denote as $w \models \varphi$).

We use $\mathcal{L}_O$ to denote the modal extension of $\mathcal{L}$ with the modalities K, N and O. We also use $\mathcal{L}_K$ to denote the modal extension of $\mathcal{L}$ with the only modality K, and $\mathcal{L}_N$ to denote the modal extension of $\mathcal{L}$ with the only modality N. We call *O-formula* a formula from $\mathcal{L}_O$ of the form $O\varphi$. Notice that, with respect to [13], we slightly change the language, using the modality K instead of B.

In the following, we call *modal atom* a sentence of the form $K\psi$, $N\psi$ or $O\psi$, with $\psi \in \mathcal{L}_O$. Given $\varphi \in \mathcal{L}_O$, we call *modal atoms of* φ (and denote as $MA(\varphi)$) the set of modal atoms occurring in φ.

The semantics of a formula $\varphi \in \mathcal{L}_O$ is defined in terms of satisfiability in a structure (w, M) where w is an interpretation (called *initial world*) and M is a set of interpretations.

Definition 1. Let w be an interpretation on $\mathcal{L}$, and let M be a set of such interpretations. We say that a formula $\varphi \in \mathcal{L}_O$ is *satisfied* in (w, M), and write $(w, M) \models \varphi$, iff the following conditions hold:

1. if $\varphi \in \mathcal{A}$, then $(w, M) \models \varphi$ iff $w(\varphi) = TRUE$;
2. $(w, M) \models \neg\varphi$ iff $(w, M) \not\models \varphi$;
3. $(w, M) \models \varphi_1 \wedge \varphi_2$ iff $(w, M) \models \varphi_1$ and $(w, M) \models \varphi_2$;
4. $(w, M) \models K\varphi$ iff for every $w' \in M$, $(w', M) \models \varphi$;
5. $(w, M) \models N\varphi$ iff for every $w' \notin M$, $(w', M) \models \varphi$;
6. $(w, M) \models O\varphi$ iff for every w', $w' \in M$ iff $(w', M) \models \varphi$.

From the above definition, it follows that the modality O can be expressed by means of the modality K and N: precisely, for each $\varphi \in \mathcal{L}_O$ and for each (w, M), $(w, M) \models O\varphi$ if and only if $(w, M) \models K\varphi \wedge N\neg\varphi$.

The above semantics is not actually the one originally proposed in [13]: in addition to the above rules, a pair (w, M) must satisfy a maximality condition for the set M, as described below (see [22] for a discussion of this issue).

In the following, $Th(M)$ denotes the set of formulae $K\varphi$ such that $\varphi \in \mathcal{L}_K$ and, for each $w \in M$, $(w, M) \models K\varphi$. Given two sets of interpretations M_1, M_2, we say that M_1 is *equivalent* to M_2 iff $Th(M_1) = Th(M_2)$. A set of interpretations M is *maximal* iff, for each set of interpretations M', if M' is equivalent to M then $M' \subseteq M$. A formula $\varphi \in \mathcal{L}_O$ is *$\mathcal{ONL}$-satisfiable* iff there exists a pair (w, M) such that $(w, M) \models \varphi$ and M is maximal.

We say that a formula $\varphi \in \mathcal{L}_O$ is *$\mathcal{ONL}$-valid* iff $\neg\varphi$ is not $\mathcal{ONL}$-satisfiable. Entailment in $\mathcal{ONL}$ is defined based on the notion of $\mathcal{ONL}$-satisfiability as follows: a formula $\varphi \in \mathcal{L}_O$ *entails* a formula $\psi \in \mathcal{L}_O$ in $\mathcal{ONL}$ iff $\varphi \supset \psi$ is $\mathcal{ONL}$-valid.

Notice that the above semantics strictly relates the logic $\mathcal{ONL}$ with modal logic K45, since there is a precise correspondence between the pairs (w, M) used in the above definition and K45 models. We recall that, with respect to the satisfiability problem, a K45 model can be considered without loss of generality as a pair (w, M), where w is a world, M is a set of worlds (possibly empty), w is connected to all the worlds in M, the worlds in M are connected with each other (i.e. M is a *cluster*), and no world in $\mathcal{M}$ is connected to w [15]. Thus, in the following we will refer to a pair (w, M) as a K45 model. Notice also that, if $\varphi \in \mathcal{L}_K$, then φ is $\mathcal{ONL}$-satisfiable if and only if it is K45-satisfiable, which is shown by the fact that, if a K45 model (w, M) satisfies such a φ, then there exists a maximal set M' equivalent to M, hence (w, M') satisfies φ. The same property holds for $\varphi \in \mathcal{L}_N$: precisely, it can be immediately proven that φ is $\mathcal{ONL}$-satisfiable if and only if $\varphi[N/K]$ is K45-satisfiable, where $\varphi[N/K]$ is the formula obtained from φ by replacing each modality N with K.

The logic of only knowing $\mathcal{OL}$ [13] simply corresponds to the fragment of $\mathcal{ONL}$ obtained by restricting the language to the subset of $\mathcal{L}_O$ not containing the modality N, i.e., built upon the modalities K and O. As a combination of the modalities K and N, the interpretation of the O modality is obtained through the maximization of the set of successors of each world satisfying an O-formula. For instance, let $\varphi \in \mathcal{L}$. Then, (w, M) is a model for $O\varphi$ iff $M = \{w : w \models \varphi\}$. Hence, prefixing φ with the modality O corresponds to maximizing the set of worlds in M, which contains all interpretations consistent with φ.

3. The tableau calculus

In this section we define a tableau calculus for $\mathcal{ONL}$. In a nutshell, the calculus is based on tableau expansion rules which include the standard rules of a tableau for the logic K45 [5, 6, 16] for handling propositional connectives and the modalities K and N: in particular, K-formulae and N-formulae in each tableau branch are handled by two separate clusters: the first cluster represents the worlds that must be connected to the initial world, i.e., the set of *reachable* worlds, while the second one represents the set of *unreachable* worlds, i.e., the worlds that must be disconnected from the initial world. In addition, a restricted cut rule enforces the presence of each modal subformula of the initial formula in each branch. Finally, a special branch closure condition enforces the restriction that each possible world must be either reachable or unreachable, i.e., must belong either to the first or to the second cluster: such a condition is decided by means of an auxiliary tableau.

and-rule: if $\sigma : \psi_1 \wedge \psi_2 \in \mathcal{B}$, and either $\sigma : \psi_1 \notin \mathcal{B}$ or $\sigma : \psi_2 \notin \mathcal{B}$, then add $\sigma : \psi_1$ and $\sigma : \psi_2$ to $\mathcal{B}$;

or-rule: if $\sigma : \psi_1 \vee \psi_2 \in \mathcal{B}$, and neither $\sigma : \psi_1 \in \mathcal{B}$ nor $\sigma : \psi_2 \in \mathcal{B}$, then add either $\sigma : \psi_1$ or $\sigma : \psi_2$ to $\mathcal{B}$;

not-and-rule: if $\sigma : \neg(\psi_1 \wedge \psi_2) \in \mathcal{B}$, and neither $\sigma : \neg\psi_1 \in \mathcal{B}$ nor $\sigma : \neg\psi_2 \in \mathcal{B}$, then add either $\sigma : \neg\psi_1$ or $\sigma : \neg\psi_2$ to $\mathcal{B}$;

not-or-rule: if $\sigma : \neg(\psi_1 \vee \psi_2) \in \mathcal{B}$, and either $\sigma : \neg\psi_1 \notin \mathcal{B}$ or $\sigma : \neg\psi_2 \notin \mathcal{B}$, then add $\sigma : \neg\psi_1$ and $\sigma : \neg\psi_2$ to $\mathcal{B}$;

not-not-rule: if $\sigma : \neg(\neg\psi) \in \mathcal{B}$ and $\sigma : \psi \notin \mathcal{B}$, then add $\sigma : \psi$ to $\mathcal{B}$.

Figure 1. Propositional tableau expansion rules.

The tableau expansion rules are reported in Figure 1 and Figure 2. The tableau calculus deals with *prefixed* formulae of the form $\sigma : \psi$, where σ is a *prefix*, i.e. either the number 0 or a pair of the form $(1, n)$ or $(2, n)$, where n is an integer greater than 0. Given a formula $\varphi \in \mathcal{L}_O$, a *branch* of the tableau for φ is a set of prefixed formulae containing the prefixed formula $0 : \varphi$ and obtained by applying the expansion rules reported in Figure 1 and Figure 2.

Let us now briefly describe the tableau rules. First, the rules reported in Figure 1 are analogous to the usual rules for handling propositional connectives in tableau methods. As for the rules reported in Figure 2 for handling modalities, the K-rule and $\neg K$-rule are standard expansion rules for the K45 modality [5]: in particular, the $\neg K$-rule adds the representation of a new world in the branch, by introducing a new prefix. The N-rule and $\neg N$-rule are exactly the same as the K-rule and $\neg K$-rule, but they affect the second cluster (i.e., the prefixes of the form $(2, n)$) instead of the first one. The $\mathcal{M}$-rule propagates each formula prefixed by a modal operator in the initial world, which simplifies the treatment of such kind of formulae by the other expansion rules.

The O-rule and $\neg O$-rule simply convert O-formulae in terms of the modalities K and N, based on the fact that $O\psi$ is equivalent to $K\psi \wedge N\neg\psi$; therefore, the presence of a formula $O\psi$ in the branch causes the addition of the formulae $K\psi$, $N\neg\psi$ in the branch, while a formula $\neg O\psi$ causes the addition of either $\neg K\psi$ or $\neg N\neg\psi$ in the branch. We remark that the presence of the O-rule and the $\neg O$-rule in our calculus is due to complexity reasons. Indeed, we could just pre-process the input formula, replacing each O-subformula by its definition in terms of K and N: however, the formula thus obtained has in general size exponential in the size of the initial formula, which leads

cut-rule: if $\psi \in MA(\varphi)$ and neither $0 : \psi \in \mathcal{B}$ nor $0 : \neg\psi \in \mathcal{B}$, then add either $0 : \psi$ or $0 : \neg\psi$ to $\mathcal{B}$;

$\mathcal{M}$-rule: if $\sigma : \mathcal{M}\psi \in \mathcal{B}$, where $\mathcal{M} \in \{K, \neg K, N, \neg N, O, \neg O\}$, and $0 : \mathcal{M}\psi \notin \mathcal{B}$, then add $0 : \mathcal{M}\psi$ to $\mathcal{B}$;

K-rule: if $0 : K\psi \in \mathcal{B}$, and there exists a prefix $(1, n)$ in $\mathcal{B}$ such that $(1, n) : \psi \notin \mathcal{B}$, then add $(1, n) : \psi$ to $\mathcal{B}$;

N-rule: if $0 : N\psi \in \mathcal{B}$, and there exists a prefix $(2, n)$ in $\mathcal{B}$ such that $(2, n) : \psi \notin \mathcal{B}$, then add $(2, n) : \psi$ to $\mathcal{B}$;

$\neg K$-rule: if $0 : \neg K\psi \in \mathcal{B}$, and there is no prefix of the form $(1, n)$ in $\mathcal{B}$ such that $(1, n) : \neg\psi \in \mathcal{B}$, then add $(1, m) : \neg\psi$ to $\mathcal{B}$, where $(1, m)$ is a new prefix in $\mathcal{B}$;

$\neg N$-rule: if $0 : \neg N\psi \in \mathcal{B}$, and there is no prefix of the form $(2, n)$ in $\mathcal{B}$ such that $(2, n) : \neg\psi \in \mathcal{B}$, then add $(2, m) : \neg\psi$ to $\mathcal{B}$, where $(2, m)$ is a new prefix in $\mathcal{B}$;

O-rule: if $0 : O\psi \in \mathcal{B}$, and either $0 : K\psi \notin \mathcal{B}$ or $0 : N\neg\psi \notin \mathcal{B}$, then add $0 : K\psi$ and $0 : N\neg\psi$ to $\mathcal{B}$;

$\neg O$-rule: if $0 : \neg O\psi \in \mathcal{B}$, and neither $0 : \neg K\psi \in \mathcal{B}$ nor $0 : \neg N\neg\psi \in \mathcal{B}$, then add either $0 : \neg K\psi$ or $0 : \neg N\neg\psi$ to $\mathcal{B}$.

Figure 2. Modal tableau expansion rules.

to tableau branches of exponential length. Conversely, by using the O-rule and $\neg O$-rule for processing O-subformulae, the generation of such exponential branches is avoided.

Finally, the cut-rule implements a restricted form of cut. Specifically, it enforces the presence in $\mathcal{B}$ of each modal subformula occurring in φ or its negation. As we shall see, such a rule is required in order to easily verify (through an auxiliary tableau) whether the set of reachable and unreachable worlds from the initial world is the set of all possible worlds, which corresponds to checking whether each world satisfies the constraints, contained in $\mathcal{B}$, concerning either the first or the second cluster.

We now define the notions of closure and completeness of a branch. We say that a branch $\mathcal{B}$ is *completed* if no expansion rule is applicable to $\mathcal{B}$. As for the notion of closure of a branch, we remark that a completed branch identifies a part of a model for ψ, in the sense that formulae of the form $0 : \psi$ in $\mathcal{B}$ are constraints for the initial world (0) of the model, while a formula of the form $(1, n) : \psi$ is a constraint for a world (n) which can be reached from the initial world, and a formula of the form $(2, m) : \psi$ is a constraint for a world (m) which *cannot* be reached from the initial world. Therefore, since the set of reachable and unreachable worlds from the initial world must be the set of all possible worlds, we have to verify that such constraints allow any world to be either reachable or unreachable. As the following example shows, this is not always the case.

Example 1. Let $\varphi = (c \vee K(a \vee b)) \wedge (Na \vee K\neg b)$. It is easy to verify that the following is a completed branch of the tableau for φ:

$$\begin{aligned}\mathcal{B} \;=\; \{&0\!:\!(c \vee K(a \vee b)) \wedge (Na \vee K\neg b),\; 0\!:\!c \vee K(a \vee b),\; 0\!:\!Na \vee K\neg b,\\ &0\!:\!K(a \vee b),\; 0\!:\!Na,\; 0\!:\!\neg K\neg b,\; (1,1)\!:\!b,\; (1,1)\!:\!a \vee b,\; (1,1)\!:\!a\}\end{aligned}$$

Due to the presence of $K(a \vee b)$ in $\mathcal{B}$, any model satisfying $\mathcal{B}$ is such that each world which can be reached from the initial world satisfies either a or b, hence all worlds that satisfy both $\neg a$ and $\neg b$ are not reachable; on the other hand, the presence of Na implies that each unreachable world satisfies a. Consequently, each interpretation satisfying both $\neg a$ and $\neg b$ is neither reachable nor unreachable according to the formulae in $\mathcal{B}$. □

In order to verify that a branch $\mathcal{B}$ allows any world to be either reachable or unreachable from the initial world, we define an auxiliary tableau for $\mathcal{B}$. The auxiliary tableau uses as expansion rules only the propositional expansion rules reported in Figure 1.

Definition 2. A branch $\mathcal{B}'$ of the auxiliary tableau for $\mathcal{B}$ is a collection of formulae (without prefix) containing:

1. each formula ψ such that $0 : \psi \in \mathcal{B}$ and either $\psi \in MA(\varphi)$ or $\psi = \neg\psi'$ and $\psi' \in MA(\varphi)$;
2. the formula $\bigvee_{0:K\psi\in\mathcal{B}} \neg\psi$ if there is at least one formula of the form $0 : K\psi$ in $\mathcal{B}$, false otherwise;
3. the formula $\bigvee_{0:N\psi\in\mathcal{B}} \neg\psi$ if there is at least one formula of the form $0 : N\psi$ in $\mathcal{B}$, false otherwise,

and obtained by applying the expansion rules reported in Figure 1.

Notice that any such branch may contain modal formulae, however they are treated as propositional atoms, i.e., they are not further analyzed.

A branch $\mathcal{B}'$ of the auxiliary tableau is completed if no propositional expansion rule is applicable to $\mathcal{B}'$, and is open if and only if (i) there is no pair of formulae in $\mathcal{B}'$ of the form ψ and $\neg\psi$; (ii) the formula false

does not belong to $\mathcal{B}'$. The auxiliary tableau for $\mathcal{B}$ is *open* if it has at least one open completed branch, otherwise it is *closed*.

Informally, the auxiliary tableau for a branch $\mathcal{B}$ of the initial tableau tries to identify a world which cannot be neither reachable nor unreachable from the initial world of any model consistent with the branch $\mathcal{B}$. If no such world exists (i.e., the tableau is closed), then the branch $\mathcal{B}$ is open, since it identifies a model for the initial formula φ.

Example 2. We now present an auxiliary tableau for the tableau of the formula φ of Example 1. Let us start from the following branch $\mathcal{B}$ (reported in Example 1) of the tableau for φ:

$$\mathcal{B} = \{0 : K(a \vee b),\ 0 : Na,\ 0 : \neg K\neg b,\ (1,1) : b,\ (1,1) : a \vee b,\ (1,1) : a\}$$

According to Definition 2, the auxiliary tableau for $\mathcal{B}$ starts with the following formulae: $\{K(a \vee b), Na, \neg K\neg b, \neg(a \vee b), \neg a\}$. By applying the expansion rules of Figure 1, we obtain the following (unique) completed branch $\mathcal{B}'$:

$$\mathcal{B}' = \{K(a \vee b),\ Na,\ \neg K\neg b,\ \neg(a \vee b),\ \neg a,\ \neg b\}$$

Such a branch is open, and identifies a world (containing both $\neg a$ and $\neg b$) which is neither reachable nor unreachable from the initial world of any model consistent with the prefixed formulae in the branch $\mathcal{B}$ of the initial tableau for φ. We thus conclude that the branch $\mathcal{B}$ is not consistent with the condition that each world must be either reachable or unreachable from the initial world. □

We now state the closure conditions for a branch of the initial tableau.

Definition 3. A completed branch $\mathcal{B}$ for $\varphi \in \mathcal{L}_O$ is *open* if and only if each of the following conditions holds:

1. there is no pair of prefixed formulae in $\mathcal{B}$ of the form $\sigma : \psi$ and $\sigma : \neg\psi$;
2. there exists no prefix σ such that the formula $\sigma : \mathsf{false}$ belongs to $\mathcal{B}$;
3. the auxiliary tableau for $\mathcal{B}$ is closed, i.e., no world is neither reachable nor unreachable.

The tableau for φ is *open* if it has at least one open completed branch, otherwise it is *closed*.

Example 3. We now apply the above defined tableau method to the formula $\varphi = c \wedge (K(a \vee b) \wedge (Oa \vee N\neg b))$. It is immediate to see that each branch of the tableau for φ contains the following set of signed formulae S, obtained by two applications of the and-rule of Figure 1 to the initial formula $0 : c \wedge (K(a \vee b) \wedge (Oa \vee N\neg b))$:

$$S = \{0 : c \wedge (K(a \vee b) \wedge (Oa \vee N\neg b)),\ 0 : c,\ 0 : K(a \vee b),\ 0 : Oa \vee N\neg b\}$$

By applying the rules reported in Figure 1 and Figure 2, we obtain the following completed branches of the tableau for φ:

$$\begin{aligned}
\mathcal{B}_1 &= S \cup \{0 : Oa,\ 0 : Ka,\ 0 : N\neg a,\ 0 : N\neg b\} \\
\mathcal{B}_2 &= S \cup \{0 : Oa,\ 0 : Ka,\ 0 : N\neg a,\ 0 : \neg N\neg b,\ (2,1) : b,\ (2,1) : \neg a\} \\
\mathcal{B}_3 &= S \cup \{0 : N\neg b, 0 : \neg Oa, 0 : \neg Ka, (1,1) : \neg a, (1,1) : a \vee b, (1,1) : a\} \\
\mathcal{B}_4 &= S \cup \{0 : N\neg b, 0 : \neg Oa, 0 : \neg Ka, (1,1) : \neg a, (1,1) : a \vee b, (1,1) : b\} \\
\mathcal{B}_5 &= S \cup \{0 : N\neg b,\ 0 : \neg Oa,\ 0 : \neg N\neg a,\ (2,1) : a,\ (2,1) : \neg b\}
\end{aligned}$$

E.g., branch $\mathcal{B}_1$ is obtained by applying the or-rule to $0 : Oa \vee N\neg b$ and choosing Oa, then applying the O-rule, thus adding $0 : Ka$, $0 : N\neg a$, and finally applying the cut-rule to $N\neg b$ choosing $0 : N\neg b$.

It is immediate to see that branch $\mathcal{B}_3$ is closed, since both $(1,1) : \neg a$ and $(1,1) : a$ belong to $\mathcal{B}_3$. For each remaining branch, we have to construct its auxiliary tableau in order to determine whether such a branch is open or closed. We obtain the following starting set of formulae $Aux(\mathcal{B}_i)$ for the auxiliary tableau of each branch $\mathcal{B}_i$:

$$\begin{aligned}
Aux(\mathcal{B}_1) &= \{K(a \vee b),\ Oa,\ Ka,\ N\neg a,\ N\neg b,\ \neg a \vee \neg(a \vee b),\ a \vee b\} \\
Aux(\mathcal{B}_2) &= \{K(a \vee b),\ Oa,\ Ka,\ N\neg a,\ \neg N\neg b,\ \neg a \vee \neg(a \vee b),\ a\} \\
Aux(\mathcal{B}_4) &= \{K(a \vee b),\ N\neg b,\ \neg Oa,\ \neg(a \vee b),\ b\} \\
Aux(\mathcal{B}_5) &= \{K(a \vee b),\ N\neg b,\ \neg Oa,\ \neg(a \vee b),\ b\}
\end{aligned}$$

We recall that both the formula $\neg a \vee \neg(a \vee b)$ in $Aux(\mathcal{B}_1)$ and $Aux(\mathcal{B}_2)$ and the formula $\neg(a \vee b)$ in $Aux(\mathcal{B}_4)$ and $Aux(\mathcal{B}_5)$ are obtained from point 2 of Definition 2, while $a \vee b$ in $Aux(\mathcal{B}_1)$, a in $Aux(\mathcal{B}_2)$, and b in $Aux(\mathcal{B}_4)$ and $Aux(\mathcal{B}_5)$ are obtained from point 3 of Definition 2. By applying the rules reported in Figure 1 we obtain that:

1. the auxiliary tableau for $\mathcal{B}_1$ is open, therefore $\mathcal{B}_1$ is closed;

2. the auxiliary tableau for $\mathcal{B}_2$ is closed, due to the presence of the formula a and the fact that each possible expansion of the formula $\neg a \vee \neg(a \vee b)$ causes the addition of $\neg a$. Hence, $\mathcal{B}_2$ is open;

3. the auxiliary tableaux for $\mathcal{B}_4$ and $\mathcal{B}_5$ are closed, due to both the presence of the formula b and the fact that the expansion of the

formula $\neg(a \vee b)$ causes the addition of the formula $\neg b$. Therefore, $\mathcal{B}_4$ and $\mathcal{B}_5$ are open.

Consequently, the tableau for φ is open. □

4. Soundness and completeness

In this section we prove soundness and completeness of the tableau calculus defined in the previous section. Specifically, we first provide an alternative semantics for $\mathcal{ONL}$ (yet equivalent to the one given in Section 2), based on Kripke structures in which unreachable worlds are explicitly represented; then, we prove completeness of our calculus with respect to such a semantics, and finally we show soundness of the method.

We start by relating satisfiability in $\mathcal{ONL}$ with the problem of finding Kripke structures constituted by an initial world and two clusters. Such structures correspond to the "extended situations" defined in [9].

Let w be a world and M, M' be sets of worlds. Then, we define satisfiability of a formula $\varphi \in \mathcal{L}_O$ in the structure (w, M, M') as follows:

1. if $\varphi \in \mathcal{L}$, then $(w, M, M') \models \varphi$ iff $w(\varphi) = \mathit{TRUE}$;
2. $(w, M, M') \models \neg\varphi$ iff $(w, M, M') \not\models \varphi$;
3. $(w, M, M') \models \varphi_1 \wedge \varphi_2$ iff $(w, M, M') \models \varphi_1$ and $(w, M, M') \models \varphi_2$;
4. $(w, M, M') \models K\varphi$ iff for every $w' \in M$, $(w', M, M') \models \varphi$;
5. $(w, M, M') \models N\varphi$ iff for every $w' \in M'$, $(w', M, M') \models \varphi$;
6. $(w, M, M') \models O\varphi$ iff for every $w' \in M$, $(w', M, M') \models \varphi$, and for every $w' \in M'$, $(w', M, M') \models \neg\varphi$.

From the above definition and Definition 1, it immediately follows that a formula $\varphi \in \mathcal{L}_O$ is $\mathcal{ONL}$-satisfiable iff there exists a structure (w, M, M') such that:

1. $(w, M, M') \models \varphi$;
2. $M \cup M'$ is the set of all possible worlds;
3. $M \cap M' = \emptyset$.

Notably, it can be proven that the last condition is not necessary, hence the existence of a structure (w, M, M') which satisfies φ and such that $M \cup M'$ is the set of all possible worlds is sufficient to establish $\mathcal{ONL}$-satisfiability of φ, as stated by the following property.

Lemma 1. Let $\varphi \in \mathcal{L}_O$. Then, φ is $\mathcal{ONL}$-satisfiable iff there exists a structure (w, M, M') such that (i) $(w, M, M') \models \varphi$ and (ii) $M \cup M'$ is the set of all possible worlds.

Proof. The proof is an immediate consequence of [9, Theorem 2.3]. □

Roughly speaking, the above property is due to the fact that the alphabet of propositions $\mathcal{A}$ is infinite, which guarantees that a finite formula cannot identify a world, i.e., no formula is able to impose the presence of a given world w in one of the two clusters M, M'.

We now prove completeness of the tableau calculus.

In the following, we call a branch $\mathcal{B}$ *weakly open* if and only if there is no pair of prefixed formulae in $\mathcal{B}$ of the form $\sigma : \psi$ and $\sigma : \neg\psi$, and there exists no prefix σ such that the formula $\sigma : \mathsf{false}$ belongs to $\mathcal{B}$. That is, we weaken the notion of open branch by discarding condition 3 in Definition 3.

Definition 4. A structure (w, M, M') is *consistent* with a completed and weakly open branch $\mathcal{B}$ if the following conditions hold:

1. for each formula ψ such that $0 : \psi \in \mathcal{B}$, $(w, M, M') \models \psi$;
2. for each prefix of the form $(1, n)$ in $\mathcal{B}$ there exists a world w' in M such that, for each formula $(1, n) : \psi \in \mathcal{B}$, $(w', M, M') \models \psi$;
3. for each prefix of the form $(2, n)$ in $\mathcal{B}$ there exists a world w' in M' such that, for each formula $(2, n) : \psi \in \mathcal{B}$, $(w', M, M') \models \psi$.

We now prove that the existence of a completed and weakly open branch in the tableau for φ corresponds to the existence of a structure (w, M, M') satisfying φ.

Lemma 2. Let $\varphi \in \mathcal{L}_O$. Then, there exists a completed and weakly open branch $\mathcal{B}$ of the tableau for φ iff there exists a structure (w, M, M') such that $(w, M, M') \models \varphi$. Moreover, (w, M, M') is consistent with $\mathcal{B}$.

Proof. The proof is obtained by a straightforward extension of the soundness and completeness proof of the tableau method for the logic K45 presented in [16]. Indeed, our tableau calculus can be seen as obtained by extending the one presented in [16] with the N-rule, the $\neg N$-rule, the O-rule, the $\neg O$-rule, and the cut-rule. Since the first four rules extend the method in order to treat the modality N in a way identical to the modality K, while the cut-rule does not affect correctness of the method, it is immediate to extend the above mentioned proof to our calculus. □

Therefore, the existence of a completed and weakly open branch implies the existence of a structure (w, M, M') satisfying φ. However, this is not enough to imply that φ is $\mathcal{ONL}$-satisfiable. From Lemma 1, we have to verify that the structure is such that $M \cup M'$ is the set of all possible worlds. To this aim, we give some auxiliary definitions.

Definition 5. Let S be a set of modal atoms. We say that a pair of sets of worlds (M, M') *induces* the partition (P, N) on S if, for each modal atom $\xi \in S$, $\xi \in P$ iff, for each world w, $(w, M, M') \models \xi$.

In the following, we say that an occurrence of a formula ψ in a formula $\varphi \in \mathcal{L}_O$ is *strict* if it is not in the scope of a modal operator. Moreover, we say that a formula $\varphi \in \mathcal{L}_O$ has *modal depth* i if each occurrence of a formula in φ lies within the scope of at most i modalities, and there is an occurrence of a formula in φ which lies within the scope of exactly i modalities.

Definition 6. Let $\varphi \in \mathcal{L}_O$ and let P, N be sets of modal atoms such that $P \cup N \supseteq MA(\varphi)$ and $P \cap N = \emptyset$. We denote with $\varphi(P, N)$ the propositional formula obtained from φ by substituting each strict occurrence in φ of a formula in P with true, and each strict occurrence in φ of a formula in N with false.

Given a branch $\mathcal{B}$ and a formula $\psi \in \mathcal{L}_O$, we denote as $(P_\mathcal{B}, N_\mathcal{B})$ the partition defined as follows:

$$\begin{aligned} P_\mathcal{B} &= \{\xi \mid 0 : \xi \in \mathcal{B} \text{ and } \xi \text{ is a modal atom}\} \\ N_\mathcal{B} &= \{\xi \mid 0 : \neg\xi \in \mathcal{B} \text{ and } \xi \text{ is a modal atom}\} \end{aligned}$$

Definition 7. Let $\varphi \in \mathcal{L}_O$ and let $\mathcal{B}$ be a completed and weakly open branch of the tableau for φ. We denote as $f_K(\mathcal{B})$ and $f_N(\mathcal{B})$ the following formulae:

$$f_K(\mathcal{B}) = \bigwedge_{0:K\psi\in\mathcal{B}} \psi(P_\mathcal{B}, N_\mathcal{B}) \qquad f_N(\mathcal{B}) = \bigwedge_{0:N\psi\in\mathcal{B}} \psi(P_\mathcal{B}, N_\mathcal{B})$$

It is immediate to verify that, due to the cut-rule, $f_K(\mathcal{B})$ and $f_N(\mathcal{B})$ are both propositional formulae.

Example 4. Let $\varphi = c \wedge (K(a \vee b) \wedge (Oa \vee N\neg b))$. As shown in Example 3, a completed and weakly open branch of the tableau for φ is the following:

$$\begin{aligned} \mathcal{B}_2 = \{&0 : c \wedge (K(a \vee b) \wedge (Oa \vee N\neg b)),\ 0 : c,\ 0 : K(a \vee b),\ 0 : Oa \vee N\neg b, \\ &0 : Oa,\ 0 : Ka,\ 0 : N\neg a,\ 0 : \neg N\neg b,\ (2,1) : b,\ (2,1) : \neg a\} \end{aligned}$$

Then, by applying the previous definitions we obtain $f_K(\mathcal{B}_2) = (a \vee b) \wedge a$ and $f_N(\mathcal{B}_2) = \neg a$. □

Lemma 3. Let $\varphi \in \mathcal{L}_O$, let w be a world, let (M, M') be sets of interpretations, and let (P, N) be the partition induced by M on a set of modal atoms S. Then, $(w, M, M') \models \varphi$ iff $(w, M, M') \models \varphi(P, N)$.

Proof. Follows immediately from Definition 6 and Definition 1. □

We now associate each completed and weakly open branch $\mathcal{B}$ with a particular structure (w, M, M') which we call *maximal model* for $\mathcal{B}$.

Definition 8. Let $\varphi \in \mathcal{L}_O$ and let $\mathcal{B}$ be a completed and weakly open branch of the tableau for φ. We call *maximal model for* $\mathcal{B}$ the structure (w, M, M'), where: (i) w is the world such that, for each propositional symbol a, $w \models a$ iff $0 : a \in \mathcal{B}$; (ii) $M = \{w \mid w \models f_K(\mathcal{B})\}$; (iii) $M' = \{w \mid w \models f_N(\mathcal{B})\}$.

Then, we prove that φ holds in the maximal model associated with a completed and weakly open branch $\mathcal{B}$ of the tableau for φ. First, we prove an auxiliary lemma.

Lemma 4. Let $\varphi \in \mathcal{L}_O$ and suppose $\mathcal{B}$ is a completed and weakly open branch of the tableau for φ. Let (w, M, M') be the maximal model for $\mathcal{B}$. Then, $(P_\mathcal{B}, N_\mathcal{B})$ is the partition induced by (M, M') on $MA(\varphi)$.

Proof. By definition of $(P_\mathcal{B}, N_\mathcal{B})$ and Definition 5, we have to prove that, for each modal atom $\xi \in MA(\varphi)$, $(w, M, M') \models \xi$ iff $0 : \xi \in \mathcal{B}$. The proof is by induction on the modal depth of modal atoms in $MA(\varphi)$.

The base case is for modal atoms of depth 1. Consider a modal atom of the form $K\psi$ such that $\psi \in \mathcal{L}$, and first suppose $0 : K\psi \in \mathcal{B}$. By Definition 7, since the propositional formula $f_K(\mathcal{B}) \supset \psi$ is a tautology, and since by Definition 8, for each $w' \in M$, $w' \models f_K(\mathcal{B})$, it follows that, for each $w' \in M$, $w' \models \psi$, hence $(w, M, M') \models K\psi$. Now suppose $0 : K\psi \notin \mathcal{B}$: then, since $\mathcal{B}$ is completed, by the cut-rule it follows that $0 : \neg K\psi \in \mathcal{B}$. Since $\mathcal{B}$ is weakly open, it follows that the propositional formula $f_K(\mathcal{B}) \supset \psi$ is not a tautology, hence there exists a world $w' \in M$ such that $w' \not\models \psi$, consequently $(w, M, M') \not\models K\psi$. In the same way, it can be shown that, if $N\psi \in MA(\varphi)$ and $\psi \in \mathcal{L}$, then $(w, M, M') \models N\psi$ iff $0 : N\psi \in \mathcal{B}$, which immediately implies that, if $O\psi \in MA(\varphi)$ and $\psi \in \mathcal{L}$, then $(w, M, M') \models O\psi$ iff $0 : O\psi \in \mathcal{B}$.

For the inductive step, suppose that, for each modal atom $\xi \in MA(\varphi)$ of modal depth less or equal to i, $\xi \in P_\mathcal{B}$ iff $(w, M, M') \models \xi$. Therefore, $(P_\mathcal{B}, N_\mathcal{B})$ agrees with the partition induced by (M, M') on all modal atoms of depth less or equal to i. Consider a modal atom $K\psi$ of $MA(\varphi)$ of modal depth $i + 1$. Since by Definition 6 the value of the sentence $\psi(P_\mathcal{B}, N_\mathcal{B})$ only depends on the value of the modal atoms of modal depth less or equal to i in $(P_\mathcal{B}, N_\mathcal{B})$, by the induction hypothesis it follows that $(w, M, M') \models \psi$ iff $(w, M, M') \models \psi(P_\mathcal{B}, N_\mathcal{B})$. Now, if $0 : K\psi \in \mathcal{B}$, then the propositional formula $f_K(\mathcal{B}) \supset \psi(P_\mathcal{B}, N_\mathcal{B})$ is a tautology, hence for each $w' \in M$, $w' \models \psi(P_\mathcal{B}, N_\mathcal{B})$, which implies that

$(w, M, M') \models K\psi$. Conversely, if $0 : K\psi \notin \mathcal{B}$, then $0 : \neg K\psi \in \mathcal{B}$, and since $\mathcal{B}$ is weakly open, it follows that the propositional formula $f_K(\mathcal{B}) \supset \psi(P_{\mathcal{B}}, N_{\mathcal{B}})$ is not a tautology, hence there exists a world $w' \in M$ such that $w' \not\models \psi(P_{\mathcal{B}}, N_{\mathcal{B}})$, which implies that $(w, M, M') \models K\psi$. In the same way, it can be shown that, if $N\psi \in MA(\varphi)$ and $N\psi$ has modal depth $i+1$, then $(w, M, M') \models N\psi$ iff $0 : N\psi \in \mathcal{B}$, which in turn implies that, if $O\psi \in MA(\varphi)$ and $O\psi$ has modal depth $i+1$, then $(w, M, M') \models O\psi$ iff $0 : O\psi \in \mathcal{B}$. □

Lemma 5. Let $\varphi \in \mathcal{L}_O$ and suppose $\mathcal{B}$ is a completed and weakly open branch of the tableau for φ. Let (w, M, M') be the maximal model for $\mathcal{B}$. Then, $(w, M, M') \models \varphi$.

Proof. Since by Lemma 4 $(P_{\mathcal{B}}, N_{\mathcal{B}})$ is the partition induced by (M, M') on $MA(\varphi)$, it follows from Lemma 3 and the fact that $\varphi(P_{\mathcal{B}}, N_{\mathcal{B}})$ is a propositional formula, that $(w, M, M') \models \varphi$ iff $w \models \varphi(P_{\mathcal{B}}, N_{\mathcal{B}})$. Now, since by Definition 8, if $0 : a \in \mathcal{B}$ then $w \models a$, and since $\mathcal{B}$ is a completed and weakly open branch, it follows that $w \models \varphi(P_{\mathcal{B}}, N_{\mathcal{B}})$, which proves the thesis. □

We now turn our attention to the auxiliary tableau of a completed and weakly open branch $\mathcal{B}$.

Lemma 6. Let $\varphi \in \mathcal{L}_O$ and let $\mathcal{B}$ be a completed and weakly open branch of the tableau for φ. Then, there exists an open branch $\mathcal{B}'$ of the auxiliary tableau for $\mathcal{B}$ iff there exists a world w such that $w \not\models f_K(\mathcal{B})$ and $w \not\models f_N(\mathcal{B})$.

Proof. Since modal atoms are treated as atomic symbols in the auxiliary tableau, the proof follows immediately from the definition of auxiliary tableau and from soundness and completeness, with respect to propositional satisfiability, of the auxiliary tableau calculus. □

The above lemma allows us to prove the key property that the maximal model for a completed open branch is a structure of the form (w, M, M') such that $M \cup M'$ is the set of all worlds.

Lemma 7. Let $\varphi \in \mathcal{L}_O$ and suppose $\mathcal{B}$ is a completed and open branch of the tableau for φ. Let (w, M, M') be the maximal model for $\mathcal{B}$. Then, $(w, M, M') \models \varphi$ and $M \cup M'$ is the set of all worlds.

Proof. The fact that $(w, M, M') \models \varphi$ follows from Lemma 5. Moreover, from Lemma 6, there is no world w such that $w \not\models f_K(\mathcal{B})$ and $w \not\models f_N(\mathcal{B})$. Therefore, from Definition 8 it follows that $M \cup M'$ is the set of all worlds. □

Therefore, we are able to state completeness of the tableau calculus.

Theorem 1. Let $\varphi \in \mathcal{L}_O$. If there exists a completed and open branch $\mathcal{B}$ of the tableau for φ, then φ is an $\mathcal{ONL}$-satisfiable formula.

Proof. Follows from Lemma 7 and Lemma 1. □

As for soundness of the tableau calculus, we start by proving the following property.

Lemma 8. Let $\varphi \in \mathcal{L}_O$, let $\mathcal{B}$ be a completed and weakly open branch of the tableau for φ, and let (w, M, M') be a structure consistent with $\mathcal{B}$. Then, for each $w' \in M$, $w' \models f_K(\mathcal{B})$, and for each $w' \in M'$, $w' \models f_N(\mathcal{B})$.

Proof. We first prove that, for each modal atom ξ such that $\xi \in MA(\varphi)$, $(w, M, M') \models \xi(P_{\mathcal{B}}, N_{\mathcal{B}})$ iff $0 : \xi \in \mathcal{B}$. The proof is by induction on the structure of the modal atoms in $MA(\varphi)$. First, consider a modal atom $\xi \in MA(\varphi)$ of modal depth 1, and suppose $0 : \xi \in \mathcal{B}$. Then, since (w, M, M') is consistent with $\mathcal{B}$, $(w, M, M') \models \xi$. Conversely, suppose $0 : \xi \notin \mathcal{B}$. Then, since $\mathcal{B}$ is completed, $0 : \neg\xi \in \mathcal{B}$, and since (w, M, M') is consistent with $\mathcal{B}$, $(w, M, M') \models \neg\xi$, consequently $(w, M, M') \not\models \xi$.

Now suppose that, for each modal atom $\xi \in MA(\varphi)$ of modal depth less or equal to i, $(w, M, M') \models \xi$ iff $0 : \xi \in \mathcal{B}$. Therefore, $(P_{\mathcal{B}}, N_{\mathcal{B}})$ agrees with the partition induced by (M, M') on all modal atoms of depth less or equal to i. Consider a modal atom ξ of $MA(\varphi)$ of modal depth $i + 1$. Since by Definition 6 the value of the sentence $\xi(P_{\mathcal{B}}, N_{\mathcal{B}})$ only depends on the value of the modal atoms of modal depth less or equal to i in $(P_{\mathcal{B}}, N_{\mathcal{B}})$, by the induction hypothesis it follows that $(w, M, M') \models \xi$ iff $(w, M, M') \models \xi(P_{\mathcal{B}}, N_{\mathcal{B}})$. Now, if $0 : \xi \in \mathcal{B}$, then, since (w, M, M') is consistent with $\mathcal{B}$, $(w, M, M') \models \xi$; conversely, if $0 : \xi \notin \mathcal{B}$, then, since $\mathcal{B}$ is completed, $0 : \neg\xi \in \mathcal{B}$, and since (w, M, M') is consistent with $\mathcal{B}$, $(w, M, M') \models \neg\xi$, consequently $(w, M, M') \not\models \xi$. Then, $(w, M, M') \models \xi$ iff $0 : \xi \in \mathcal{B}$, and since $(w, M, M') \models \xi$ iff $(w, M, M') \models \xi(P_{\mathcal{B}}, N_{\mathcal{B}})$, the thesis follows.

Now, since for each modal atom $\xi \in MA(\varphi)$, $(w, M, M') \models \xi(P_{\mathcal{B}}, N_{\mathcal{B}})$ iff $0 : \xi \in \mathcal{B}$, it follows that $(w, M, M') \models K\psi(P_{\mathcal{B}}, N_{\mathcal{B}})$ for each $0 : K\psi \in \mathcal{B}$, hence by Definition 7 it follows that, for each $w' \in M$, $w' \models f_K(\mathcal{B})$. In the same way, since $(w, M, M') \models N\psi(P_{\mathcal{B}}, N_{\mathcal{B}})$ for each $0 : N\psi \in \mathcal{B}$, Definition 7 implies that, for each $w' \in M'$, $w' \models f_N(\mathcal{B})$. □

We are now ready to prove soundness of our method.

Theorem 2. Let $\varphi \in \mathcal{L}_O$ be an $\mathcal{ONL}$-satisfiable formula. Then, there exists a completed and open branch $\mathcal{B}$ of the tableau for φ.

Proof. From Lemma 1, it follows that there exists a structure (w, M, M') such that $(w, M, M') \models \varphi$ and $M \cup M'$ is the set of all worlds. Moreover, by Lemma 2, there exists a completed and weakly open branch $\mathcal{B}$ of the tableau for φ such that (w, M, M') is consistent with $\mathcal{B}$, and by Lemma 8, for each $w' \in M$, $w' \models f_K(\mathcal{B})$, and for each $w' \in M'$, $w' \models f_N(\mathcal{B})$. Since $M \cup M'$ is the set of all worlds, it follows that there is no world w'' such that $w'' \not\models f_K(\mathcal{B})$ and $w'' \not\models f_N(\mathcal{B})$, hence by Lemma 6 the auxiliary tableau for $\mathcal{B}$ is closed, which implies that $\mathcal{B}$ is open. □

Therefore, from Theorem 1 and Theorem 2 we are able to prove correctness of our tableau calculus.

Theorem 3. Let $\varphi \in \mathcal{L}_O$. Then, φ is $\mathcal{ONL}$-satisfiable iff there exists a completed open branch of the tableau for φ.

5. Complexity analysis

In this section we analyze the computational aspects of reasoning in $\mathcal{ONL}$, based on our tableau calculus. We start by briefly introducing the complexity classes mentioned in the following (refer e.g. to [12] for further details). All the classes we use reside in the *polynomial hierarchy.* In particular, the complexity class Σ_2^p is the class of problems that are solved in polynomial time by a nondeterministic Turing machine that uses an NP-oracle (i.e., that solves in constant time any problem in NP), and Π_2^p is the class of problems that are complement of a problem in Σ_2^p. It is generally assumed that the polynomial hierarchy does not collapse, and that a problem in the class NP is computationally easier than a Σ_2^p-hard or Π_2^p-hard problem.

We now analyze the complexity of our tableau method.

Theorem 4. The $\mathcal{ONL}$-satisfiability problem is in Σ_2^p.

Proof. It is immediate to verify that: (i) any (prefixed) formula appearing either in a branch of the initial tableau for φ or in a branch of an auxiliary tableau has size linear in the size of φ; (ii) any completed branch of the initial tableau for φ contains a polynomial number (in the size of φ) of prefixed formulae; (iii) any completed branch of the auxiliary tableau for a branch $\mathcal{B}$ contains a number of formulae which is polynomial in the size of $\mathcal{B}$; (iv) for each tableau expansion rule, both deciding whether the rule can be applied and applying the rule can be done in polynomial time. Therefore, each completed branch of the auxiliary tableau for a branch $\mathcal{B}$ can be constructed in time polynomial in the size of φ. Consequently, from Theorem 3 the tableau method is able to check whether a completed branch $\mathcal{B}$ of the tableau for φ is open in nondeterministic polynomial time. Moreover, each completed branch of the tableau for φ can also be constructed in polynomial time. Thus, our method can be seen as a nondeterministic procedure which is able to decide in polynomial time whether there exists an open branch of the tableau for φ, using an NP-oracle for deciding whether a completed branch is open. □

Theorem 5. The $\mathcal{ONL}$-satisfiability problem is Σ_2^p-complete.

Proof. Membership in Σ_2^p follows from previous theorem. As for Σ_2^p-hardness, it has been proven in [21] that deciding satisfiability of a formula in $\mathcal{OL}$ is a Σ_2^p-complete problem. Therefore, it immediately follows that satisfiability of a formula in $\mathcal{ONL}$ is Σ_2^p-hard. □

Therefore, the above theorem states that adding a "knowing at most" modality N to the logic of only knowing $\mathcal{OL}$ does not increase the computational complexity of reasoning.

As immediate corollaries of the above property, we obtain that both validity and entailment in $\mathcal{ONL}$ are Π_2^p-complete problems (see definition of entailment in $\mathcal{ONL}$ given in Section 2). Notice that, since the satisfiability problem in $\mathcal{ONL}$ is Σ_2^p-hard, a *single* tableau for such a logic should have branches of exponential length (unless $\Sigma_2^p = \mathrm{NP}$). Instead, as shown above, by using two distinct tableaux we are able to decide satisfiability using polynomial space.

Finally, the analysis of our method also points out that satisfiability in $\mathcal{ONL}$ can be decided in nondeterministic polynomial time, if the construction of auxiliary tableaux can be avoided. As an immediate consequence of Definition 2, it follows that the construction of the auxiliary tableau for a branch $\mathcal{B}$ is only needed when $\mathcal{B}$ contains at least one formula of the form $0 : K\psi$ and at least one formula of the form $0 : N\psi$ (otherwise the auxiliary tableau is closed, due to the presence of the formula false in each branch of the auxiliary tableau). Therefore, those branches in which there are either no formulae of the form $0 : K\psi$ or no formulae of the form $0 : N\psi$ should be generated first, which can be realized by avoiding, whenever possible, the addition of modal subformulae of the form $K\psi$, $N\psi$, $O\psi$ in the branch, by a suitable application of branching rules, in particular the or-rule, not-and-rule, and cut-rule. Observe that a branch can contain either no formulae of the form $0 : K\psi$ or no formulae of the form $0 : N\psi$ even if the initial formula contains occurrences of both K-subformulae and N-subformulae.

6. Related work and conclusions

The tableau calculus presented in this paper allows for establishing the computational complexity of reasoning about only knowing and knowing at most: in particular, we have proven that satisfiability in the modal propositional fragment of $\mathcal{ONL}$ is Σ_2^p-complete, while validity is Π_2^p-complete. Hence, our tableau calculus is the first "optimal" method for reasoning about only knowing and knowing at most.

Several studies have recently proposed tableau (or related) calculi for nonmonotonic reasoning. E.g., tableaux [20, 19] and sequent calculi [1] have been proposed for circumscription and minimal model reasoning, and analogous methods have been defined for default logic [23, 2]. Moreover, tableaux for nonmonotonic modal logics have been presented, in particular for autoepistemic logic [18] and for both McDermott and Doyle's and ground nonmonotonic modal logics [4]. None of such methods is able to deal with reasoning about only knowing and knowing at most, since no embedding is known of the logic $\mathcal{ONL}$ (or even the logic $\mathcal{OL}$) into another nonmonotonic formalism.

On the other hand, it is well-known [3] that the logic $\mathcal{ONL}$ is able to naturally embed some of the major nonmonotonic formalisms, i.e., autoepistemic logic, prerequisite-free default logic, (disjunctive) logic programming under the stable model semantics, and circumscription. Due to such embeddings, our method can be also seen as a general, semantic-based tableau calculus which is able to uniformly cover a large family of nonmonotonic formalisms.

Two possible developments of the present work are worth mentioning. The first one concerns the analysis of reasoning about only knowing and knowing at most in a first-order setting: in particular, it should be interesting to see whether it is possible to extend the tableau calculus presented in this paper for the modal propositional case to (a fragment of) the first-order modal language. Another interesting development is the extension of the tableau method to the multi-agent generalization of $\mathcal{ONL}$ [9, 7]. Notably, with respect to the single-agent case, such an extension requires to face further semantical and computational issues, since different alternative semantics have been proposed for the multi-agent extension of $\mathcal{ONL}$, and it has been shown [9] that reasoning in each of such semantics is PSPACE-hard.

References

1. Bonatti, P. A. and N. Olivetti: 1997a, 'A Sequent Calculus for Circumscription'. In: *Proc. of CSL'97*, Vol. 1414 of *LNAI*. pp. 98–114.

2. Bonatti, P. A. and N. Olivetti: 1997b, 'A Sequent Calculus for Skeptical Default Logic'. In: *Proc. of TABLEAUX'97*, Vol. 1227 of *LNAI*. pp. 107–121.
3. Chen, J.: 1994, 'The logic of only knowing as a unified framework for non-monotonic reasoning'. *Fundamenta Informaticae* **21**, 205–220.
4. Donini, F. M., F. Massacci, D. Nardi, and R. Rosati: 1996, 'A Uniform Tableaux Method for Nonmonotonic Modal Logics'. In: J. J. Alferes, L. M. Pereira, and E. Orlowska (eds.): *Proc. of JELIA'96*, Vol. 1126 of *LNAI*. pp. 87–103.
5. Fitting, M.: 1983, *Proof Methods for Modal and Intuitionistic Logics*. Reidel.
6. Goré, R.: 1991, 'Semi-analytic Tableaux for Modal Logics with Application to Nonmonotonicity'. *Logique et Analyse* **133-134**.
7. Halpern, J. Y.: 1997, 'A theory of knowledge and ignorance for many agents'. *Journal of Logic and Computation* **7**(1), 79–108.
8. Halpern, J. Y. and G. Lakemeyer: 1995, 'Levesque's axiomatization of only knowing is incomplete'. *Artificial Intelligence* **74**(2), 381–387.
9. Halpern, J. Y. and G. Lakemeyer: 1996, 'Multi-agent only knowing'. In: *Proc. of TARK'96*.
10. Halpern, J. Y. and Y. Moses: 1985, 'Towards a theory of knowledge and ignorance: Preliminary report'. In: K. Apt (ed.): *Logic and models of concurrent systems*. Springer-Verlag.
11. Halpern, J. Y. and Y. Moses: 1992, 'A Guide to Completeness and Complexity for Modal Logics of Knowledge and Belief'. *Artificial Intelligence* **54**, 319–379.
12. Johnson, D. S.: 1990, 'A Catalog of Complexity Classes'. In: J. van Leuven (ed.): *Handbook of Theoretical Computer Science*, Vol. A. Elsevier, Chapt. 2.
13. Levesque, H. J.: 1990, 'All I Know: a Study in Autoepistemic Logic'. *Artificial Intelligence* **42**, 263–310.
14. Lifschitz, V.: 1994, 'Minimal belief and negation as failure'. *Artificial Intelligence* **70**, 53–72.
15. Marek, W. and M. Truszczyński: 1993, *Nonmonotonic Logics – Context-Dependent Reasoning*. Springer-Verlag.
16. Massacci, F.: 2000, 'Single step tableaux for modal logics'. *Journal of Automated Reasoning* **21**(4).
17. Moore, R. C.: 1985, 'Semantical Considerations on Nonmonotonic Logic'. *Artificial Intelligence* **25**, 75–94.
18. Niemelä, I.: 1988, 'Decision procedure for autoepistemic logic'. In: *Proc. of CADE'88*, Vol. 310 of *LNCS*. pp. 675–684.
19. Niemelä, I.: 1996, 'A Tableau Calculus for Minimal Model Reasoning'. In: *Proc. of TABLEAUX'96*, Vol. 1071 of *LNAI*. pp. 278–294.
20. Olivetti, N.: 1992, 'Tableaux and Sequent Calculus for Minimal Entailment'. *Journal of Automated Reasoning* **9**, 99–139.
21. Rosati, R.: 2000a, 'On the decidability and complexity of reasoning about only knowing'. *Artificial Intelligence* **116**, 193–215.
22. Rosati, R.: 2000b, 'Tableau calculus for only knowing and knowing at most'. *Proc. of TABLEAUX-2000*, Vol. 1847 of *LNAI*, pp. 383–397.
23. Schwind, C. B.: 1990, 'A Tableaux-based Theorem Prover for a Decidable Subset of Default Logic'. In: *Proc. of CADE'90*, Vol. 449 of *LNCS*. pp. 528–542.