Exercise 1 Denoting $L(s) = G(s)P(s)$, in the Laplace domain the error and output evolutions are described by

$$e(s) = W_e(s)v(s), \quad y(s) = W(s)v(s)$$

with $W(s) = \frac{L(s)}{1+L(s)}$ and $W_e(s) = \frac{1}{1+L(s)}$. We shall split the controller $G(s) = G_2(s)G_1(s)$ with $G_1(s)$ designed for steady-steady specifications (i.e., (ii)) and $G_2(s)$ for the remaining ones.

(ii) As $e(s) = W_e(s)v(s)$, one needs $W_e(0) = 0$ and $\left| \frac{\partial W_e}{\partial s} \right|_{s=0} \leq M$ with $M = 0.02$. As the plant possesses a pole at $s = 0$, the former condition (i.e., $W_e(0) = 0$) is verified whereas the latter one is ensured by the inequality above

$$\left| \frac{W_e(s)}{s} \right|_{s=0} \leq M.$$

Setting $G_1(s) = k_1$ and, for the time-being $G_2(s) = 1$, one has that the specification is fulfilled by setting

$$\left| \frac{1}{5k} \right| \leq 0.02 \implies |k_1| \geq 10.$$

Accordingly, we fix $k_1 = 10$ while constraining the gain of the outer loop of the controller to verify $G_2(0) > 1$.

(iii)-(iv) By inspecting the Bode plots (Figure 1) of

$$L_1(s) = \frac{50}{s(s+1)} \quad \text{(1)}$$

one notes that as $\omega \in [8, 14] \text{ rad/s}$

1. the magnitude is decreasing and $|L_1(j\omega)|_{dB} \in [-11.88, -2.21]$;
2. the phase is decreasing and $\angle L_1(j\omega) \in [-175.91^\circ, -172.88^\circ] \text{ rad/s}$.

Accordingly, for increasing the phase margin $m_\phi \geq 50^\circ$, an anticipative action is needed with phase contribution of at least $45.91^\circ$ at $\omega = 14 \text{ rad/sec}$ and $42.88^\circ$ at $\omega = 8 \text{ rad/s}$. Moreover, specification (iv) sets a bound over the magnitude of the controller so that one has

$$|G(j\omega)|_{dB} = |G_1(j\omega)|_{dB} + |G_2(j\omega)|_{dB} = |k_1|_{dB} + |G_2(j\omega)|_{dB} \leq 36$$

implying

$$|G_2(j\omega)|_{dB} \leq 16 \quad \text{and} \quad |G_2(0)|_{dB} > 1$$

where the latter bound comes from specification (ii). Accordingly, we set the outer control loop as composed of a one anticipating action and one proportional term so
Figure 1: Bode plots of (1)

Getting

\[ G_2(s) = k_2 G_a(s), \quad G_a(s) = \frac{1 + \tau_a s}{1 + \frac{\tau_a s}{m_a}s}. \]

\[ |k_2|_{dB} + \max_{\omega \geq 0} |G_a(j\omega)|_{dB} \leq 16 \quad \text{and} \quad |k_2|_{dB} > 0. \]

For saving the control effort of the controller, we shall set an anticipative function labeled by \( m_a = 6 \) and acting at \( \omega_n = 2 \text{ rad/s} \) (that is \( \tau_a = \frac{2}{\omega_n} \)) so getting that \( \angle G_a(j8) = 45^\circ \), \( |G_a(j8)|_{dB} \approx 6.53 \), \( \max_{\omega \geq 0} |G_a(j\omega)|_{dB} \approx 16 \). Accordingly, we shall set \( \omega_t^* \in [8, 14] \text{ rad/s} \) in such a way that

\[ |L_1(j\omega_t^*)|_{dB} - 6.53 = 0 \]

that is achieved for \( \omega_t^* \approx 10.3 \text{ rad/sec} \) in correspondence of which \( \angle L_1(j\omega) \approx -174.44^\circ \). Thus, setting \( k_2 = 1 \) specifications (iii) – (iv) are satisfied with \( m_{\phi}^* \approx 50.6^\circ \) as confirmed
by the Bode plots of

\[ L(s) = G(s)P(s) = 10 \frac{1 + \frac{\tau_a s}{m_a} s}{1 + \frac{\tau_a s}{m_a} s(s + 1)} \]  

reported in Figure 2.

![Bode Diagram](image_url)

**Figure 2: Bode plots of (3)**

(i) The Nyquist plot of the open loop system

\[ L(s) = kG_a(s)P(s) = 10 \frac{1 + 0.7939 s + 1}{1 + 0.1323 s^3} \]  

are reported in Figure 3. The number of counter-clockwise encirclements of \(-1 + j0\) on behalf of the extended Nyquist plot of \(L(j\omega)\) is 0 as the number the open loop poles of \(L(s)\) with positive real part. Thus, the system is asymptotically stable in closed loop.

**Exercise 2** Denoting \(L(s) = G_2(s)P(s)\) with \(P(s) = P_1(s)P_2(s)\), one has that in the Laplace domain the inputs-output evolutions are described by

\[ y(s) = W_d(s)d(s) + W(s)v(s), \quad W_d(s) = \frac{1 - G_1(s)P_2(s)}{1 + L(s)}, \quad W(s) = \frac{L(s)}{1 + L(s)}. \]
(i) For ensuring that \( y(t) \equiv 0 \) for all disturbances, one has to define \( G_1(s) \) in such a way that \( W_d(s) = 0 \). This is achieved by setting \( G_1(s)P_2(s) = 1 \) that is

\[
G_1(s) = \frac{s - 2}{s + 4}.
\]

(ii) As \( e(s) = W_e(s)v(s) \) with \( W_e(s) = \frac{1}{1 + L(s)} \), for ensuring that \( |e(t)| \leq 1 \) for ramp references \( v(t) = 1 \), one needs

\[
W_e(0) = 0, \quad \left| \frac{\partial W_e}{\partial s} \right|_{s=0} \leq 1
\]

so that one needs \( G_2(s) = \frac{k_1}{s} \hat{G}_2(s) \) with \( k_1 \in \mathbb{R} \). Setting for the time-being \( \hat{G}_2(s) = 1 \), one gets that the specification is verified if \( k_1 \) is such that

\[
\left| \frac{W_e(s)}{s} \right|_{s=0} \leq 1
\]

that is implied by \( k_1 \geq \frac{1}{2} \). Accordingly, we fix \( k_1 = \frac{1}{2} \) and set \( \hat{G}_2(0) \geq 1 \).

For stabilizing the closed-loop system (that is assigning all poles of \( W(s) \) with negative real part) setting \( \hat{G}_2(s) = \hat{k} \) with \( \hat{k} > 1 \) is not enough. Indeed, setting

\[
F(s) = \frac{\hat{k}}{2s}P(s) = \frac{\hat{k}}{2} \frac{s + 4}{s(s - 1)(s - 2)}
\]

Figure 3: Nyquist plot of (3)
one has that the closed-loop pole polynomial is $p_F(s, \hat{k}) = s^3 - 3s^2 + (2 + k)s + 4k$ exhibiting two sign variations in the coefficients for all $\hat{k} \in \mathbb{R}$. Accordingly, by invoking the necessary condition of the Routh criterion, there exists no $\hat{G}(s) = \hat{k} \in \mathbb{R}$ making the closed-loop system asymptotically stable. Thus, one can set

$$\hat{G}(s) = \frac{s + z}{s + p}$$

with $p \in \mathbb{R}$ chosen in such a way that

$$\frac{-p + 3 + 4 + z}{2} < 0 \implies p - z > 7.$$  \hfill (5)

Hence, setting $z = 4$, the above necessary condition is satisfied for $p = 21$. Thus, one has that

$$L(s) = \hat{k} \frac{(s + 4)^2}{s(s - 1)(s - 2)(s + 21)}$$

so that the stabilizing $\hat{k}$ can be computed by invoking the Routh criterion and computing the Routh table associated to the closed-loop pole polynomial

$$p_L(s, k) = s^4 + 18s^3 + (k - 61)s^2 + (8k + 42)s + 16k$$

that is given by

\[
\begin{array}{ccc}
 r^4 & 1 & k - 61 & 16k \\
 r^3 & 9 & 4k + 21 \\
 r^2 & \frac{5k - 570}{9} & 16k \\
 r^1 & \frac{20k^2 - 2319k - 11970}{5k - 570} \\
 r^0 & k \\
\end{array}
\]

Thus, one has that for $k > 120.9$ (and thus $\hat{k} > 241.8$) the closed-loop system is asymptotically stable under the controller

$$G_2(s) = \frac{\hat{k} \cdot s + 4}{2 s(s + 21)}.$$  

The root locus of $P(s)G_1(s)G_2(s) = \frac{\hat{k} \cdot s + 4}{2 s(s - 1)(s + 21)}$ is equivalent to the one of

$$K(s) = \frac{s + 4}{s(s - 1)(s + 21)}.$$  

Accordingly, denoting by $n$ and $m$ the number of poles and zeros and $r = n - m = 2$ as the relative degree, one has that positive locus possesses one vertical asymptote centered at

$$s_0 = \frac{-21 + 1 + 4}{2} = -8.$$  

Moreover, the positive locus possesses one singularity of order $\mu = 2$ at $(s^*, \hat{k}^*) \approx (0.48, 1.2)$
Figure 4: Root Locus of \( P(s)G_1(s)G_2(s) = \frac{k}{2} \frac{s+4}{s(s-1)(s+21)} \).

as solutions to the equalities

\[
p_K(s, \tilde{k}) = s^3 + 20s^2 + (\tilde{k} - 21)s + 4\tilde{k} = 0
\]

\[
\frac{\partial p_K(s, \tilde{k})}{\partial s} = 3s^2 + 40s + \tilde{k} - 21.
\]

The locus is reported in Figure 4.

**Exercise 3** The closed-loop denominator of the input-output transfer function is given by

\[ p(s) = NUM(1 + P(s)) = s^3 + 3s^2 + (3 + z)s + 1 - z. \]

(i) For the roots of \( p(s) \) to possess real part smaller or equal to \(-\frac{1}{2}\) it is necessary and sufficient, by the Routh criterion, that the polynomial

\[ p_1(s) = p(s - \frac{1}{2}) = s^3 + 3s^2 + (z + \frac{3}{4})s - \frac{3z}{2} + \frac{1}{8} \]

is Hurwitz. By computing the Routh table

| \( r^3 \) | 1 | \( z + \frac{3}{4} \) |
| \( r^2 \) | \( \frac{3}{2} \) | \( -\frac{3z}{2} + \frac{1}{8} \) |
| \( r^1 \) | 3z + 1 |
| \( r^0 \) | \( -3z + \frac{1}{4} \) |

the closed-loop system has all poles with real part smaller than \(-\frac{1}{2}\) for all \( z \in (-\frac{1}{3}, \frac{1}{12}) \).

(ii) It is evident that for \( z = 0 \), the roots of \( S^* := \{ s \in \mathbb{C} \text{ s.t. } p(s) = 0 \} = \{-1\} \). For
determining, more in general, all roots of \( p(s) \) it is enough for the discriminant of the polynomial \( p(s) \) given by

\[
\Delta^* = -4z^2(z + 27)
\]

to be non-negative. This is the case for \( z \in \{0\} \cup (-\infty, -27) \)