Exercise 1 Denoting $L(s) = G(s)P(s)$, in the Laplace domain the error and output evolutions are described by

$$e(s) = W_e(s)v(s), \quad y(s) = W(s)v(s) + W_d(s)d(s)$$

with $W(s) = \frac{L(s)}{1+L(s)}$, $W_e(s) = \frac{1}{1+L(s)}$ and $W_d(s) = \frac{P(s)}{1+L(s)}$. We shall split the controller $G(s) = G_2(s)G_1(s)$ with $G_1(s)$ designed for steady-steady specifications (i.e., $(ii) - (iii)$) and $G_2(s)$ for the remaining ones.

(ii) As $e(s) = W_e(s)v(s)$, one needs $W_e(0) = 0$. As the plant possesses a pole at $s = 0$, the former condition (i.e., $W_e(0) = 0$) is already verified by the plant.

(iii) For $y(t) \equiv 0$ under constant disturbances, one integrator (that is a pole in zero) is needed right before the entering point of the disturbance so to guarantee $W_d(0) = 0$. As the integrating action of the plant is located after the disturbance action, a further integrator needs to be included through the inner controller $G_1(s)$; accordingly, one sets

$$G_1(s) = \frac{1}{s}.$$ 

(iv) As the inner control action $G_1(s)$ has dimension $n_1$, the outer feedback loop is constrained to be of at most one-dimensional. Consequently, for maximizing the cross-over frequency $\omega^* \phi$ while making the phase margin $m^* \phi \geq 50^\circ$, $G_2(s)$ is composed of at most one anticipative action plus one proportional term; namely, it is of the form

$$G_2(s) = kG_a(s), \quad G_a(s) = \frac{1 + \tau_a s}{1 + \frac{\tau_a}{m_a} s}.$$ 

By inspecting the Bode plots (Figure 1) of

$$L_1(s) = G_1(s)P(s) = \frac{1}{s^2(s+1)} \quad (1)$$

one notes that it is clear that $\angle L_1(j\omega) \in [-180^\circ, -270^\circ]$. As a consequence, let us set the attenuating function so to induce the largest phase contribution; namely, one sets $\omega_n = \omega^* \tau_a = 4$ rad/s with $m_a = 16$ in such to increase the phase (at the working frequency $\omega^* \phi$) of $\approx 61.9275^\circ$ with a magnitude contribution of $\approx 12.0412$ dB. At this point, for making $m^* \phi \geq 50^\circ$ with such an action, $\omega^* \phi$ needs to be chosen so that the following inequality is satisfied

$$180^\circ + \angle L_1(j\omega^* \phi) + 61.9275^\circ \geq 50^\circ \implies \angle L_1(j\omega^* \phi) \geq -191.9275^\circ.$$ 

Also, for guaranteeing that the closed-loop system is asymptotically stable it is necessary to guarantee that the extended Nyquist plot of $L(s) = G_2(s)L_1(s)$ makes no encirclements of $-1 + j0$. Since $\angle L_1(j\omega) \in [-180^\circ, -270^\circ]$ and we do want to maximize $\omega^* \phi$, it is convenient to set $\omega^* \phi$ as the frequency corresponding to $\angle L_1(j\omega^* \phi) \geq -191.9275^\circ$.
that is $\omega_t^* \approx 0.2112 \text{ rad/s}$. As a consequence, one has $\tau_a = \frac{\omega_n}{\omega_t^*} \approx 18.9363$. At this point, the gain $k \in \mathbb{R}$ must be set so to make $\omega_t^*$ the new cross-over frequency, that is to guarantee

$$|k|_{dB} + |G_a(j\omega_t^*)|_{dB} + |L_1(j\omega_t^*)|_{dB} = 0 \implies |k|_{dB} \approx -38.8639$$

so finally getting $k \approx 0.0113$. Thus, the new open-loop system

$$L(s) = G_2(s)L_1(s) = 0.0113 \frac{1 + 18.9363s}{1 + 1.1837s} \frac{1}{s^2(s + 1)}$$

satisfies specification $(iii)$ as confirmed by Figure 2.

(i) The Nyquist plot of the open loop system

$$L(s) = kG_a(s)P(s) = 10 \frac{1 + 0.7939s}{1 + 0.1323s} \frac{s + 1}{s^3}$$

are reported in Figure 3. The number of counter-clockwise encirclements of $-1 + j0$ on behalf of the extended Nyquist plot of $L(j\omega)$ is 0 as the number the open loop poles of $L(s)$ with positive real part. Thus, the system is asymptotically stable in closed loop.
Exercise 2 As by assigning all poles in \(-1\) one is also making the closed-loop system asymptotically stable, specification (i) is satisfied as a byproduct of (ii). Accordingly, we shall set \(G(s) = G_2(s) = G_1(s)\).

As the center of the asymptotes of the root locus associated to \(P(s)\) is positive, it is straightforward to state that a static feedback \(G(s) = k\) can neither stabilize the system or, especially, assign all poles with positive real part. Thus, for stabilizing the system the controller needs to possess at least one pole for moving the center of the asymptotes and increase the relative degree to \(r = 2\) (that is \(G(s) = k \frac{1}{s + p}, p > 0\)).

Also, note that because the plant \(P(s)\) possesses two zeros at \(-1\) (the desired location for the closed-loop poles), it might be convenient to cancel them under feedback in such a way that a finite (not high) gain is admissible for fulfilling the specification. Moreover, by drawing the root locus of \(\frac{1}{(s+1)^2}\) it is evident that \(\bar{G}(s)\) needs to be set in such a way to create a singularity at \(s = -1\). Thus, we assume the feedback of the form

\[
G(s) = \frac{as^3 + bs^2 + cs + d}{(s + 1)^2(es + f)}.
\]
and thus

\[ L(s) = \frac{as^3 + bs^2 + cs + d}{(s - 1)(s - 2)(s - 4)(es + f)}. \]

Denoting \( p(s) \) as the denominator of the closed-loop input-output transfer function \( W(s) = \frac{L(s)}{1 + L(s)} \)

\[ p(s) = (s - 1)(s - 2)(s - 4)(es + f) + as^3 + bs^2 + cs + d \]

one now proceeds by looking at \( a, b, c, d, e, f \in \mathbb{R} \) such that

\[ p(s) = (s + 1)^4 \]

that is

\[ es^4 + (a - 7e + f)s^3 + (b + 14e - 7f)s^2 + (c - 8e + 14f)s + d - 8f = s^4 + 4s^3 + 6s^2 + 4s + 1 \]

Thus, by equating the terms with the same power of \( s^i \) \( (i = 0, 1, \ldots, 4) \) one gets

\[ a = 11, \quad b = -8, \quad c = 12, \quad d = 1, \quad e = 1 \quad \text{and} \quad f = 0 \]
The system possesses one pseudo-periodical mode associated to the eigenvalues 

\[ \lambda = \frac{-k}{2} \pm i \frac{\sqrt{3}}{2} k. \]

The transfer function associated to the system is given by

\[ P(s) = \frac{1}{s^2 + ks + k^2} \]

so that for \( u(t) = u_1(t) + u_2(t) \) with \( u_1(t) = 1_+ \) and \( u_2(t) = \sin_+(t) \), the forced output response can be computed in the Laplace domain as

\[ y(t) = y_1(t) + y_2(t), \quad y_i(t) = \mathcal{L}^{-1}(P(s)u_i(s))[t], \quad u_i(s) = \mathcal{L}(u_i(t))[s] \]

for \( i = 1, 2 \).
Accordingly, by computing
\[
y_1(s) = P(s)u_1(s) = \frac{1}{s^2 + ks + k^2} \cdot s = \frac{1}{k^2} s - \frac{s + k}{k^2 s^2 + ks + k^2} = \frac{1}{k^2} s - \frac{s + k}{k^2 (s + \frac{k}{2})^2 + \frac{3}{4} k^2} - \frac{\sqrt{3} k}{2 (s + \frac{k}{2})^2 + \frac{3}{4} k^2}
\]
\[
y_2(s) = P(s)u_2(s) = \frac{1}{s^2 + ks + k^2} \cdot \frac{1}{s^2 + 1} = \frac{As + B}{s^2 + 1} + \frac{C(s + \frac{k}{2}) + D}{s^2 + ks + k^2}
\]
with
\[
A = -\frac{k}{k^4 - k^2 + 1}, \quad B = \frac{k^2 - 1}{k^4 - k^2 + 1}, \quad C = \frac{k}{k^4 - k^2 + 1}, \quad D = -\frac{k^2 - 2}{2(k^4 - k^2 + 1)}
\]
one gets
\[
y_1(t) = \frac{1}{k^2} e^{-\frac{k}{2} t} \cos \left(\frac{\sqrt{3}}{2} kt\right) - \frac{1}{\sqrt{3} k^2} e^{-\frac{k}{2} t} \sin \left(\frac{\sqrt{3}}{2} kt\right) + 
\]
\[
y_2(t) = A \sin t + B \cos t + C e^{-\frac{k}{2} t} \cos \left(\frac{\sqrt{3}}{2} kt\right) + \frac{2\sqrt{3}}{3} D e^{-\frac{k}{2} t} \sin \left(\frac{\sqrt{3}}{2} kt\right)
\]
If the system is asymptotically stable (that is if, and only if, $k > 0$), the steady-state response is given by

$$y_{ss}(t) = \frac{1}{k^2} + A\sin t + B\cos t.$$

The corresponding transient response gets the form

$$y_{tran}(t) = y(t) - y_{ss}(t) = \left(\frac{1}{k^2} + C\right)e^{-\frac{k}{2}t}\cos\left(\frac{\sqrt{3}}{2}kt\right) + \frac{\sqrt{3}}{3}\left(\frac{1}{k^2} + 2D\right)e^{-\frac{k}{2}t}\sin\left(\frac{\sqrt{3}}{2}kt\right).$$

As a requirement, the transient response must be bounded, for a suitable $k > 0$, as follows

$$|y_{tran}(t)| \leq 0.05, \quad \forall t \geq T_a = 10^{-2}.$$

Accordingly, denoting $R := \left|\frac{1}{k^2} + C\right| + \frac{\sqrt{3}}{3}\left|\frac{1}{k^2} + 2D\right|$, one has that

$$|y_{tran}(t)| \leq Re^{-\frac{k}{2}t}$$

so that $k$ must verify

$$e^{\frac{k}{2}T_a} \geq \frac{20}{R} \implies k \geq \frac{2}{T_a}\ln 20R.$$

The above condition can be qualitatively checked for temptative values of $k \in \mathbb{R}$. As an example, the requirement is fulfilled for $k = 10$. 