Exercise 1 We have in Laplace domain

\[ Y(s) = \frac{P(s)}{1 + L(s)} d(s) + \frac{L(s)}{1 + L(s)} v(s), \quad L(s) = G(s)P(s), \]

so that \( W_d(s) = \frac{P(s)}{1 + L(s)} \) (disturbance-to-output transfer function) and \( W_e(s) = \frac{1}{1 + L(s)} \) (input-to-error transfer function).

(i) Since there is no integral action before the entering point of \( d \), we set \( G(s) = \frac{1}{s} \hat{G}(s) \) so that the steady state response with constant disturbances is

\[ y_0 = W_d(0) = 0 \]

(ii),(iii) From (ii) we have the following constraint on \( \hat{G}(s) : |\hat{G}(j\omega)| \leq 36 \text{ dB} \) for all \( \omega \). The Bode plots of \( \hat{P}(s) = \frac{1}{s} P(s) \) are drawn in Fig. 1.

We have form the Bode plots in Fig. 1

\[ |\hat{P}(j5)|_{dB} \approx -27.8 dB, \quad \text{Arg}(\hat{P}(j5)) \approx -191^\circ \]
\[ |\hat{P}(j10)|_{dB} \approx 40 dB, \quad \text{Arg}(\hat{P}(j10)) \approx -180^\circ \]

Let us place the new crossover frequency \( \omega_\ast \) at 5 rad/sec with the desired phase margin \( \geq 30^\circ \), using \( \hat{G}(s) \) and recalling that we must satisfy \( \text{vert}\hat{G}(j\omega)| \leq 36 \text{ dB} \) for all \( \omega \). For doing this, \( \hat{G}(s) \) must be such that

\[ |\hat{G}(j5)|_{dB} \approx 27.8 dB, \quad \text{Arg}(\hat{G}(j5)) \approx 42^\circ \]

Let

\[ \hat{G}(s) = KR_a(s) = K \frac{1 + \tau a s}{1 + m a s} \]

and choose (from the compensating functions Bode plots) \( m_a = 6, \omega_N = 2 \text{ rad/sec} \) with \( \omega_\ast^\omega = 5 \). At \( \omega_N = 2 \text{ rad/sec} \) we have magnitude increase equal to 6 dB and phase increase equal to 45\(^\circ\). For \( \omega_\ast^\omega = 5 \) we obtain \( 2 = \omega_N = \omega_\ast^\omega \tau_a = 5\tau_a \Rightarrow \tau_a = 2/5 \).

We have \( |R_a(j\omega_\ast^\omega)P(j\omega_\ast^\omega)| = -27.8 + 6dB = -21.8dB \) and \( \text{Arg}(\cdot) \approx -191^\circ + 45^\circ = 146^\circ \) which would imply a phase margin \( \approx 34^\circ \geq 30^\circ \)(as required by (iii)). For having an overall magnitude increase of 27.8dB at \( \omega_\ast^\omega = 5 \text{ rad/sec} \) we choose a proportional action \( K = 21.8dB \) so that to have \( \omega_\ast^\omega \approx 5 \text{ rad/sec} \). Our controller \( G(s) \) is finally

\[ \hat{G}(s) = 12.28 \frac{1 + \frac{2}{5} s}{1 + \frac{1}{15} s} \]
The Bode plots of $G(s)P(s)$ and its Nyquist plot are drawn in Fig. 2 and 3. The Nyquist plot shows that the closed-loop system is asymptotically stable (we have $-1 + 1 = 0$ counterclockwise tours around the point $-1 + 0j$).

**Exercise 2** We have in Laplace domain

\[
Y(s) = \frac{L_1(s)}{1 + L_2(s)}d_1(s) + \frac{1}{P(s)} \frac{L_1(s)}{1 + L_2(s)}d_2(s) + \frac{L_2(s)}{1 + L_2(s)}m(s)
\]

where $L_1(s) = \frac{P(s)}{1 + P(s)}$ and $L_2(s) = G(s)L_1(s)$.

(ii) Since the $d_1$ to $y$ transfer function is $W_{d_1}(s) = \frac{L_1(s)}{1 + L_2(s)}$, we must have for unit ramp disturbance $d_1$

\[
\left| \frac{1}{s} \frac{L_1(s)}{1 + L_2(s)} \right|_{s=0} \leq 0.1 \Rightarrow \left| \frac{NUM(G(s))}{DEN(sG(s))} \right|_{s=0} \leq 0.1
\]
which implies that $G(s) = \frac{K_{G,1}}{s}G_2(s)$ with

$$|K_{G,1}| \geq 10.$$  

Choose $|K_{G,1}| = 10$.

(iii) Since the $d_1$ to $y$ transfer function is $W_{d_2}(s) = \frac{1}{P(s)} \frac{L_1(s)}{1+L_2(s)}$ we must have for constant disturbance $d_2$

$$\left. \frac{1}{P(s)} \frac{L_1(s)}{1+L_2(s)} \right|_{s=0} = 0$$

which is true thanks to the pole at $s = 0$ in $G(s)$.

(i) Recall that $G(s)$ is required to be two-dimensional. Therefore, $\tilde{G}(s)$ may have the form $\frac{K_{G,2}(s+z)^2}{s+p}$ so that $G(s) = \frac{1}{5} \tilde{G}(s)$ is indeed two dimensional and realizable (two pole-zero actions plus a proportional action). The direct path transfer function is

$$L_2(s) = G(s)L_1(s) = 10 \frac{s + 2}{s^2(s-1)^2} \tilde{G}(s)$$

the first zero of $\tilde{G}(s)$ will decrease the zero-pole excess from 3 to 2 and the zero-pole action will move the asymptote center to the left: the new asymptote center will be required to satisfy

$$s'_0 = \frac{4 - p + 2z}{2} < -1$$

Moreover, notice that the zeroes of $L_2(s)$ must be all with real part $< -1$ (in such a way that by increasing the gain the closed-loop poles will move to the left of $\text{Re}(s) = -1$). We choose $z = 3$ and $p = 20$. Next, we choose $K_{G,2}$ from the Routh table of $NUM(1 + G(s)P(s))|_{s=-1} = s^5 + 13s^4 + (K - 10)s^3 + (5K + 235)s^2(8K - 224)s + 4K + 76$. We obtain as first column of
Figure 4: Positive root locus of \( P(s) = \frac{(s+1)^2}{(s-1)(s^2+1)} \)

Figure 5: Negative root locus of \( P(s) = \frac{(s+1)^2}{(s-1)(s^2+1)} \)

the Routh table

\[
\begin{array}{cccc}
1 & & & \\
13 & & & \\
2K_{G_2} - 62 & & & \\
10K_{G_2}^2 - 165K_{G_2} - 4859 & & & \\
2(K_{G_2} - 31) & & & \\
18K_{G_2}^3 - 839K_{G_2}^2 + 404K_{G_2} + 256733 & & & \\
10K_{G_2}^2 - 165K_{G_2} - 4859 & & & \\
4K_{G_2} + 76 & & & \\
\end{array}
\]

which gives \( K_{G_2} > \max\{31, 19, 31.78\} = 31.78 \) for having no sign variations.
Exercise 3. (i) The zero-pole excess is \( n - m = 1 \), the asymptote center \( s_0 = 1 + 21 = 3 \) (it is not useful for \( n - m = 1 \)) and the singular points are determined by the equations:

\[
p(s, k) = (s^2 + 1)(s - 1) + K(s + 1)^2 = 0
\]
\[
\frac{d}{ds}p(s, k) = 2s^2 - 2s + s^2 + 1 + 2K(s + 1) = 0
\]

We obtain as solution \( s \approx -3.95 \). From the Routh table of \( NUM(1 + KP(s)) = s^3 + (K - 1)s^2 + (1 + 2K)s + K - 1 \) we obtain as first column

\[
\begin{align*}
1 \\
K - 1 \\
2K \\
K - 1
\end{align*}
\]

which implies \( K > 1 \) for having no sign variations. Therefore, the closed-loop system with any \( G(s) = K > 1 \) is asymptotically stable. The root locuses of \( P(s) = \frac{(s+1)^2}{s(s-1)(s^2+1)} \) have been drawn in Fig. 4 and 5.

(ii) The root locuses of \( P(s) = \frac{(s+1)^2}{s(s-1)(s^2+1)} \) have been drawn in Fig. 4 and 5. Notice the singular points \( s \approx 0.2 \pm 0.6j \) for \( K \approx 0.2 \) and \( s \approx 0.4 \) for \( K \approx 0.1 \) for the positive locus (Fig. 6) and \( s \approx -2.4 \) for \( K \approx -28 \) for the negative locus (Fig. 7). From the Routh table of \( NUM(1 + KP(s)) = s^4 - s^3 + (K + 1)s^2 + (2K - 1)s + K \) we obtain as first column

\[
\begin{align*}
1 \\
-1 \\
3K \\
2K(3K - 1) \\
K
\end{align*}
\]

which implies there is no \( G(s) = K \) for which the closed-loop system is asymptotically stable.
Exercise 4. Our process

\[
\dot{x} = Ax + Bu + \tilde{P}d, \ y = Cx
\]  

(1)

where

\[
A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \ B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \ \tilde{P} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ C = \begin{pmatrix} 1 & 0 \end{pmatrix}.
\]

We first check that \((A, B)\) is stabilizable. Indeed, it is even controllable \((R = (B \ 1AB) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}).\)

We solve the problem with the output regulation procedure. Since \(d = D\sin t\) we choose an exosystem for \(d\) of the form

\[
\dot{w}_d = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} w_d = S_d w_d
\]

whose solutions have the form

\[
\dot{w}_d = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} w_d(0)
\]

so that the disturbance is generated as \(d(t) = \begin{pmatrix} 1 & 0 \end{pmatrix} w_d(t) = Q_d d(t)\) corresponding to the initial conditions \(w_d(0) = \begin{pmatrix} 0 \\ D \end{pmatrix}\).

Since \(v = \delta_{-1}(t)\) we choose an exosystem for \(v\) of the form

\[
\dot{w}_v = 0 = S_v w_v
\]

whose solutions have the form

\[
\dot{w}_v = w_v(0)
\]
so that the reference input \(v\) is generated as \(v(t) = w_v(t) = Q_v v(t)\) corresponding to the initial conditions \(w_v(0) = 1\). In the overall, we have the exosystem

\[
\dot{w} = \begin{pmatrix} S_d & 0 \\ 0 & S_v \end{pmatrix} w = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} w
\]

and \(w = \begin{pmatrix} w_d \\ w_v \end{pmatrix}\). The output of the exosystem \(q = Qw\) for generating the vector \(\begin{pmatrix} d \\ v \end{pmatrix}\) (disturbances and reference inputs) will be

\[
q = \begin{pmatrix} d \\ v \end{pmatrix} = Qw = \begin{pmatrix} Q_d & 0 \\ 0 & Q_v \end{pmatrix} w = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} w
\]

Finally the tracking error is defined as

\[
e = y - v = Cx - Q_v w = \begin{pmatrix} 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} w
\]

The process (1) together with the exosystem becomes

\[
\begin{align*}
\dot{x} &= Ax + Bu + Pd, \\
\dot{w} &= Sw, \\
e &= Cx + Qw,
\end{align*}
\]

(2) (3)

with

\[
P = \tilde{P}Q_w = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Q = -Q_v = \begin{pmatrix} 0 & 0 & -1 \end{pmatrix}.
\]

The regulator equations to be solved foursome \(\Pi \in \mathbb{R}^{2 \times 3}\) and \(\Gamma \in \mathbb{R}^{1 \times 3}\) are

\[
\Pi S = A\Pi + B\Gamma + P
\]

\(C\Pi = Q\)

From the second equation

\[
\begin{pmatrix} 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} \pi_1 \\ \pi_2 \end{pmatrix} \Rightarrow \pi_1 = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}
\]

and using this in the first equation we get

\[
\pi_2 = \begin{pmatrix} -1 & 0 & 0 \end{pmatrix}, \quad \Gamma = \begin{pmatrix} 0 & -1 & 0 \end{pmatrix}.
\]

Therefore,

\[
\Pi = \begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}, \quad \Gamma = \begin{pmatrix} 0 & -1 & 0 \end{pmatrix}
\]
and the (state feedback) regulator is given by

\[ u = F(x - \Pi w) + \Gamma w \]

where \( F \in \mathbb{R}^{1 \times 2} \) is any matrix for which \( \sigma(A + BF) \in \mathbb{C}^- \) (use Ackermann’s formula for finding \( F: F = -\gamma p^*(A) \)). For example, with \( F = \begin{pmatrix} -1 & -2 \end{pmatrix} \) we assign the eigenvalues of \( A + BF \) both in \(-1\).