Exercise 1  Denoting $L(s) = G(s)P(s)$, one has
\[ y(s) = W(s)v(s), \quad e(s) = W_\epsilon(s)v(s) \]
with $W(s) = \frac{L(s)}{1+L(s)}$ and $W_\epsilon(s) = \frac{1}{1+L(s)}$. As usual, we shall split the controller in two loops; namely, $G(s) = G_2(s)G_1(s)$ with $G_1(s)$ designed for satisfying steady-state specifications (i.e., (ii)) whereas the outer loop $G_2(s)$ is defined for transient and stability requirements (i.e., (iii) and (i)).

(ii) Set for the time being $G_2(s) = 1$. As $e_{ss}(t) = W_\epsilon(0)t + \frac{\partial W_\epsilon}{\partial s}(0)$, when $v(t) = t$, one needs $W_\epsilon(0) = 0$ and $\frac{\partial W_\epsilon}{\partial s}(0) = 0$. Accordingly, two integrating actions are needed. As the plant itself already possesses a pole at $s = 0$, we set the inner control loop $G_1(s) = \frac{1}{s}$.

(iii) By inspecting the Bode Plots of
\[ L_1(s) = G_1(s)P(s) = \frac{1}{s^2(s-1)} \] (1)
reported in Figure 1, one notices that the outer loop control action $G_2(s)$ needs to be chosen so to

1. increase the value of the phase at $\omega_t^* = 3$ rad/s as so that $m_\varphi^* = 180^\circ + \angle G_2(j\omega_t^*) |+\angle L_1(j\omega_t^*) | \geq 30^\circ$ with $\angle L_1(j\omega_t^*) = -360^\circ + 71.57^\circ$ so implying $\angle G_2(j\omega_t^*) | \geq 138.43^\circ$.
2. decrease the magnitude at $\omega_t^* = 3$ rad/s so to guarantee $|G_2(j\omega_t^*)|_{dB} + |L_1(j\omega_t^*)|_{dB} = 0$ with $|L_1(j\omega_t^*)|_{dB} \approx -29.08$

Accordingly, as no bound is apriori set over the gain of $G_2(s)$ (that is $G_2(0)$), we shall design the outer loop as composed of anticipating actions aimed at increasing the phase at $\omega_t^* = 3$ rad/s and a gain to make $\omega_t^* = 3$ rad/s the new cross-over frequency.

Thus, the structure we propose for $G_2(s)$ is the following one
\[ G_2(s) = kG_a(s), \quad k > 0 \]
with $G_a(s) = G_a^1(s)G_a^2(s)$. In particular, we introduce 3 anticipating actions of the form
\[ G_a^1(s) = \frac{\left(1 + \frac{\tau_a^1 s}{1 + \frac{\tau_a^1}{m_a^1} s}\right)}{1 + \frac{\tau_a^1}{m_a^1} s}, \quad G_a^2(s) = \frac{1 + \frac{\tau_a^2 s}{1 + \frac{\tau_a^2}{m_a^2} s}}{1 + \frac{\tau_a^2}{m_a^2} s} \]
with
- $G_a^1(s)$ composed of two identical anticipating functions acting at $\omega_N^1 = 3$ rad/sec with $m_a^1 = 16$ (that is at $\tau_a^1 = 1$) so that $\angle G_a^1(j\omega_N^1) | \approx 121^\circ$ and $|G_a^1(j\omega_N^1)|_{dB} \approx 20.08$.
- $G_a^2(s)$ being one anticipating function with $\omega_N^2 = 5$ rad/sec and $m_a^2 = 3$ (that is $\tau_a^2 = \frac{5}{3}$) so that $\angle G_a^2(j\omega_N^2) | \approx 20^\circ$ and $|G_a^2(j\omega_N^2)|_{dB} \approx 8$.

In this way, as $k > 0$, $m_\varphi^* = 32.57^\circ$ whereas $k$ needs to be chosen so that $|k|_{dB} + |G_a(j\omega_t^*)|_{dB} = 29.08 = 0$ with $|G_a(j\omega_t^*)|_{dB} \approx 28.08$ so requiring $k = 1.124$. The bode
Figure 1: Bode plots of (1)
Figure 2: Bode plots of (2)
plots of

\[ L(s) = G_2(s)G_1(s)P(s) = 1.124 \left( \frac{1 + s}{1 + \frac{1}{16s}} \right)^2 \frac{1 + \frac{s}{2}}{1 + \frac{s}{2} s^2(s - 1)} \]  

are reported in Figure 2.

(i) The Nyquist plot of \((2)\) is reported in Figure 3. The number of counter-clockwise encirclements of \(-1 + j0\) on behalf of the extended Nyquist plot of \(L(j\omega)\) is 1 that is also coincident with the open loop poles with positive real part. Thus, the system is asymptotically stable in closed loop.

**Exercise 2 a)** Denoting by \(n\) and \(m\) the number of poles and zeros of the transfer function, the relative degree of \(P(s)\) is given by \(r = n - m = 1\). Accordingly, the root locus possesses two asymptotes centered at

\[ s_0 = \frac{1 + 5}{2} = 3 \]

that can be discarded. Introducing \(k \in \mathbb{R}\) and defining \(p(s, k) = (s^2 + 1)(s - 1) + k(s + 5)\) as the polynomial defining the closed-loop poles under \(G(s) = k\), one gets that singularities are the solutions to

\[ p(s, k) = s^3 - s^2 + (k + 1)s + 5k - 1 = 0 \]
\[ \frac{\partial p(s, k)}{\partial s} = 3s^2 - 2s + k + 1 = 0 \]

By solving the equations above, it turns out that the negative locus possesses one sin-
gularity with multiplicity $\mu = 2$ in correspondence of $(s^*, k^*) \approx (-7.7, -194.27)$. What is left to do is now to quantify the number of intersection of the root locus with the imaginary axis. Those intersecting points correspond to values of $k \in \mathbb{R}$ for which the Routh table of $p(s, k) = s^3 + (k + 9)s^2 + (4k + 14)s + 4k - 24$ is not regular. Thus, by developing computations one gets

\[
\begin{array}{c|ccc}
0 & 1 & k + 1 & 0 \\
1 & -1 & 5k - 1 & 0 \\
2 & -6k & 0 & 0 \\
3 & 5k - 1 & 0 & 0 \\
4 & 0 & 0 & 0 \\
5 & 0 & 0 & 0 \\
\end{array}
\]

The Routh table is not regular for $k = \frac{1}{5}$ and $k = 0$ so implying that the positive locus intersects the imaginary axis in correspondence of $k = \frac{1}{5}$ corresponding to the closed-loop pole $s = 0$ and at $k = 0$ corresponding to the open loop poles $s = \pm j$. The root locus is reported in Figure 4.

b) From the above root locus and the Routh table it is evident that there exists no controller $G(s) = k$ asymptotically stabilizing the feedback system.

c) For ensuring zero-steady state error to constant inputs, the controller $G(s)$ must possess a pole at $s = 0$. Thus, we set $p_1 = 0$ and, for the sake of notations, we shall denote
hereinafter $p = p_2$. Thus, by denoting

$$L(s) = k \frac{(1 + z_1 s)(1 + z_2 s)(s + 5)}{s(s^2 + 1)(s - 1)(s + p)} = k \frac{(s + \hat{z}_1)(s + \hat{z}_2)(s + 5)}{s(s^2 + 1)(s - 1)(s + p)}$$

one gets that a necessary condition for assigning the poles with real part smaller or equal than 3 is that the new center of the asymptotes satisfies

$$s_0' = \frac{3 - p + \hat{z}_1 + \hat{z}_2}{2} < -3.$$ 

Accordingly, $p$, $\hat{z}_1$ and $\hat{z}_2$ can be fixed as $p = 25$, $\hat{z}_1 = 3$ and $\hat{z}_2 = 4$ so getting $s_0' = -6$ and implying $z_1 = \frac{1}{3}$, $z_2 = \frac{1}{4}$ and $k = 12\hat{k}$. At this point, one can set $\hat{k} \in \mathbb{R}$ (or equivalently $k \in \mathbb{R}$) by invoking the extended Routh criterion. Namely, one sets $\hat{k}$ so to make the shifted closed-loop polynomial

$$p^*_L(s, \hat{k}) = p_L(s - 3, \hat{k}) = (s - 3)(s^2 - 6s + 10)(s - 4)(s + 22) + \hat{k}(s + 1)(s + 2)$$

$$= s^5 + 21s^4 + (\hat{k} - 222)s^3 + (3\hat{k} + 1266)s^2 + (2\hat{k} - 3004)s + 2640$$

Hurwitz. By computing the Routh table

<table>
<thead>
<tr>
<th>$r^5$</th>
<th>1</th>
<th>$\hat{k} - 222$</th>
<th>$2\hat{k} - 3004$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r^4$</td>
<td>$\hat{k} - 544$</td>
<td>$\hat{k} + 422$</td>
<td>880</td>
</tr>
<tr>
<td>$r^3$</td>
<td>$k^2 - 131\hat{k} - 214730$</td>
<td>$3\hat{k} - 4946$</td>
<td>880</td>
</tr>
<tr>
<td>$r^2$</td>
<td>$k - 544^2$</td>
<td>$k - 544$</td>
<td>880</td>
</tr>
<tr>
<td>$r^1$</td>
<td>$(k^3 - 2073\hat{k}^2 + 320392\hat{k} + 267210300)$</td>
<td>$(k - 544)^2$</td>
<td>880</td>
</tr>
<tr>
<td>$r^0$</td>
<td>$(k - 544)^2$</td>
<td>(880)</td>
<td></td>
</tr>
</tbody>
</table>

one gets the specification satisfied for $\hat{k} > 1815.4$.

**Exercise 3 (i)** For computing the forced response, one needs to rewrite the input

$$u(t) = \begin{cases} 
1 - e^{t-1} & \text{as } t \in [0, 1) \\
 t - 1 & \text{as } t \in [1, 2) \\
 1 & \text{as } t \geq 2.
\end{cases}$$

as the linear combination of elementary signals. Accordingly, one gets

$$u(t) = u_1(t) - e^{-1}u_2(t) - u_1(t - 1) + u_2(t - 1) + u_3(t - 1) - u_3(t - 2)$$

with

$$u_1(t) = 1_+, \quad u_2(t) = e^t_+, \quad u_3(t) = t_+.$$ 

Accordingly, as the system is time-invariant and linear, the output response can be computed as

$$y(t) = y_1(t) - e^{-1}y_2(t) - y_1(t - 1) + y_2(t - 1) + y_3(t - 1) - y_3(t - 2)$$

(3)
with
\[ y_i(t) = \mathcal{L}^{-1}(P(s)u_i(s))[t], \quad u_i(s) = L(u_i(t))[s], \quad i = 1, 2, 3. \]

In particular, one has
\[ y_1(t) = K\mathcal{L}^{-1}\left(\frac{1}{s(1+s)}\right)[t] = K\mathcal{L}^{-1}\left(\frac{1}{s}\right)[t] - K\mathcal{L}^{-1}\left(\frac{1}{1+s}\right)[t] = K(1_+ - e^{-t}) \]
\[ y_2(t) = K\mathcal{L}^{-1}\left(\frac{1}{(s-1)(s+1)}\right)[t] = \frac{K}{2}\mathcal{L}^{-1}\left(\frac{1}{s-1}\right)[t] - \frac{K}{2}\mathcal{L}^{-1}\left(\frac{1}{s+1}\right)[t] = \frac{K}{2}(e^{t} - e^{-t}) \]
\[ y_3(t) = K\mathcal{L}^{-1}\left(\frac{1}{s^2(s+1)}\right)[t] = K\mathcal{L}^{-1}\left(\frac{1}{s+1}\right)[t] - K\mathcal{L}^{-1}\left(\frac{1}{s}\right)[t] + \mathcal{L}^{-1}\left(\frac{1}{s^2}\right)[t] \]
\[ K(e^{t} - 1_+ + t_+). \]

By substituting, after suitable time-shift, the above equalities in (3) one gets the result.

(ii) The system has a well-defined steady-state response as it is asymptotically stable (all poles, that we assume also as eigenvalues, are with negative real part). The steady-state response can be computed starting from (3) by neglecting all terms whose effect is vanishing in time. Accordingly, one gets
\[ y_{ss}(t) = K. \]

(iii) The settling time is defined as the time instant \( T_s > 0 \) for which the output response remains within 5% of its steady-state values for all \( t \geq T_s \). Accordingly, by defining the transient response as \( y_{tran}(t) = y(t) - y_{ss}(t) \) one gets that \( K \) needs to be chosen so that, for \( T_s \leq 10^{-3} \) and for all \( t \geq T_s \)
\[ |y_{tran}(t)| \leq 0.95|y_{ss}(t)|. \]

By rewriting \( y_{tran}(t) = K\bar{y}_{tran}(t) \) and \( y_{ss}(t) = K\bar{y}_{ss}(t) \) one gets that the above equality is independent upon \( K \) so that it is not possible to set the gain to decrease at will the settling time.