Exercise 1 In Laplace domain the disturbance and input-to-output responses are given by

\[ y(s) = L(s)e(s) + d_2(s) + \frac{P_1(s)}{1 + L_1(s)}d_1(s) \]

with

\[ L(s) = G_1(s)P_2(s)P(s), \]
\[ L_1(s) = G_1(s)P_1(s), \]
\[ P(s) = \frac{L_1(s)}{1 + L_1(s)}. \]

In order to meet requirements (ii) and (iii) set

\[ G_1(s) = \frac{1}{s}, \quad G_2 = \frac{1}{s}G(s) \]

with one-dimensional \( G(s) \) (recall that \( G_1 \) is required to be one dimensional and \( G_2 \) two-dimensional). Therefore

\[ P(s) = \frac{2.1s + 0.1}{s^2 + 1.1s0.1}, \quad L(s) = \frac{G(s)}{s} \frac{1}{2.1s + 0.1} \frac{2.1s + 0.1}{s^2 + 1.1s0.1} = \frac{G(s)}{s} \frac{10}{s(s+1)(1+10s)}. \]

From the Bode plot of \( L(s) = P_2(s)P(s) \) (Fig. 1) we see that we have to increase the phase (to maximize the phase margin) using an anticipative+proportional action \( G(s) = KR_a(s) = K\frac{1 + \tau_a s}{1 + \tau_a m_a s} \).

In order to maximize the phase margin, we choose \( m_a = 16 \) with \( \omega_N = 4 \) rad/sec (maximum phase value) at \( \omega^* \approx 0.0001 \) rad/sec (where the Bode plot of the phase of \( L(s) \) is higher: actually, any \( \omega^* \leq 0.0001 \) is good as well). We obtain \( \tau_a = 4/0.0001 = 4000 \). Therefore, the anticipative action is \( R_a(s) = \frac{1 + 4000 s}{s(s+1)(1+10s)} \).

The controller \( G_1(s) \) is given finally by

\[ G_1(s) = \frac{2.23 \times 10^{-5} s + 4000}{s + \frac{1 + 4000}{16}}. \]

The Bode plot of \( G_1(s)L(s) \) is drawn in Fig. 3 and shows that we have a crossover frequency \( \omega_c = 10^{-3} \) rad/sec with a phase margin \( m_\phi \approx 150^\circ \). The Nyquist plot shows that the closed-loop system is asymptotically stable (we have 0 counterclockwise tours around the point \(-1 + 0j\)).

Exercise 2 (a) The root locus of \( P(s) = \frac{s^3 + 3}{s(s-3)(s+10)^2} \) is drawn in Fig. 1.
The zero-pole excess is $n - m = 3$ and the asymptote center is at $s_0 = \frac{3 - 20 + 3}{3} = -\frac{14}{3} \approx 4.67$.

The Routh table applied to $NUM(1 + KP(s)) = s^4 + 17s^3 + 40s^2 + (K - 300)s + 3K$ has the first column given by

$$
\begin{array}{c}
1 \\
17 \\
980 - K \\
-K^2 + 413K - 294000 \\
980 - K \\
51K
\end{array}
$$

The number of sign variation in this column confirms the presence of 2 closed-loop poles with positive real part for $K > 0$ and 1 closed-loop pole with positive real part for $K < 0$. Moreover, neither locus crosses the imaginary axis. Therefore, there is no $K$ such that the closed-loop system $W(s) = \frac{KP(s)}{1 + KP(s)}$ is asymptotically stable (point (b)).

(c) It is required to find one dimensional $G(s)$ such that the closed-loop system $W(s) = \frac{G(s)P(s)}{1 + G(s)P(s)}$ is asymptotically stable with poles having real part $\leq -2$ and steady state error
to unit ramp input $|e_1| \leq 0.1$. Since the asymptote center is $< -2$ and the zeroes of $P(s)$ have real part $< -2$, we have only to decrease the zero-pole excess from 3 to 2 (keeping the asymptote center $< -2$) and then increase the gain to move the poles. Let

$$G(s) = \frac{G_1(s)}{1 + Ts}$$

with $G_1(s) = K(1 + \bar{z}s)$ and $\bar{z} > 0$. Choose $K$ in such a way that, whatever $\bar{z} > 0$ is,

$$|e_1| = \left| \frac{1}{s} W_e(s) \right|_{s=0} = \left| \frac{1}{s} \frac{1}{1 + G(s)P(s)} \right|_{s=0} = \left| \frac{1}{s} \frac{1}{1 + G_1(s)P(s)} \right|_{s=0}$$

$$= \left| \frac{(s - 3)(s + 10)^2}{K(1 + \bar{z}s)(s + 3) + (s - 3)(s + 10)^2} \right|_{s=0} = \frac{100}{K} \leq 0.1$$

which gives $|K| \geq 1000$. Next, noticing that

$$P(s)G_1(s) = K\bar{z} \frac{(s + \frac{1}{\bar{z}})(s + 3)}{(s - 3)(s + 10)^2} = \bar{K} \frac{(s + \frac{1}{\bar{z}})(s + 3)}{(s - 3)(s + 10)^2}$$
with $\tilde{K} = K\bar{z}$, choose $\bar{z} > 0$ such that the new asymptote center $s'_0$ remains $< -2$

$$s'_0 = \frac{-14 + \frac{1}{\bar{z}}}{2} < -2 \Rightarrow \bar{z} > 0.1$$

Let’s try the values $\bar{z} = 1/4$ and (tentatively large) $K = 10^3$. The Routh table applied to $NUM(1+G_1(s)P(s)) = s^4 + 9s^3 + 212s^2 + 462s + 1140$ has the first column given by

1
3
$\frac{241}{3}$
1
570

which implies stability of the closed-loop $\frac{G_1(s)P(s)}{1+G_1(s)P(s)}$. However, $G_1(s)$ is not implementable as such and we have to add the pole $\frac{1}{1+Ts}$ for obtaining the implementable controller

$$G(s) = \frac{G_1(s)}{1+Ts}$$

Choose tentatively (small) $T = 10^{-4}$ and check through the Routh table, applied to $NUM(1+G(s)P(s))$, not to have sign variations in the first column.

(d) We seek a controller

$$G(s) = K\frac{(s+10)^2}{s+3} \frac{s+z}{s+p} \frac{1}{1+Ts}$$

where we are canceling as many stable poles and zeroes of $P(s)$ as possible. The closed-loop transfer function is

$$W(s) = \frac{L(s)}{1+L(s)}, \quad L(s) = K\frac{s+z}{(s-3)(s+p)(1+Ts)} = K\frac{s+z}{(s+p_1)(s+p_2)(s+p_3)(s+p_4)}$$
for $\bar{K} = \frac{K}{T}$ and for some $p_1, p_2, p_3, p_4 > 0$ such that
\[(s + p_1)(s + p_2)(s + p_3)(s + p_4) = s(s + \bar{T})(s + p)(s - 3) + \bar{K}(s + z)\]
where $\bar{T}$. In particular, we obtain by comparison from above
\[
\bar{T} + p - 3 = p_1 + p_2 + p_3 + p_4
\]
\[
\bar{T}p - 3p - 3T = p_3(p_1 + p_2) + p_1p_2 + p_4(p_1 + p_2 + p_3)
\]
\[
K - 3\bar{T}p = p_4(p_3(p_1 + p_2) + p_1p_2) + p_1p_2p_3
\]
\[
Kz = p_1p_2p_3p_4
\]
(1)

Since the output response in Laplace domain to a unit step input is
\[Y(s) = W(s)\frac{1}{s} = \bar{K}\frac{s + z}{s(s + p_1)(s + p_2)(s + p_3)(s + p_4)}\]
we obtain in time
\[y(t) = \bar{K}[R_1e^{-p_1t} + R_2e^{-p_2t} + R_3e^{-p_3t} + R_4e^{-p_4t}]\delta_{-1}(t)\]
with residuals
\[R_1 = -\frac{z - p_1}{(p_2 - p_1)(p_3 - p_1)(p_4 - p_1)p_1}, R_2 = -\frac{z - p_2}{(p_1 - p_2)(p_3 - p_2)(p_4 - p_2)p_2}\]
\[R_3 = -\frac{z - p_3}{(p_1 - p_3)(p_2 - p_3)(p_4 - p_3)p_3}, R_4 = -\frac{z - p_4}{(p_1 - p_4)(p_2 - p_4)(p_3 - p_4)p_4}\]
The steady state output response is
\[y_{ss}(t) = \bar{K}\frac{z}{p_1p_2p_3p_4}\]
The transient output response is
\[|y(t) - y_{ss}(t)| = |\bar{K}[R_1e^{-p_1t} + R_2e^{-p_2t} + R_3e^{-p_3t} + R_4e^{-p_4t}]| \leq |R_1| + |R_2| + |R_3| + |R_4|e^{-\text{min}_i p_it}\]
We require that
\[ |y(t) - y_{ss}(t)| \leq \frac{5}{100} |y_{ss}(t)|, \forall t \geq T_a = 20^{-2}. \]

We obtain the condition
\[ e^{T_a \min \ p_i} \geq 25 \frac{p_1 p_2 p_3 p_4}{z} [\left| R_1 \right| + \left| R_2 \right| + \left| R_3 \right| + \left| R_4 \right|] \]
\[ \Rightarrow T_a \min \ p_i \geq \ln(25 \frac{p_1 p_2 p_3 p_4}{z} [\left| R_1 \right| + \left| R_2 \right| + \left| R_3 \right| + \left| R_4 \right|]) \tag{2} \]

We choose (tentatively) \( p_1 = 1, p_2 = 2, p_3 = 4 \) and \( p_4 = 25 \) so that from (1)
\[ \bar{T} = -3.5, \ p = 38.5, \ K = -354.25, \ z = -0.0677 \]

and from (2) we finally get the sought value of \( T_a \):
\[ T_a = \ln\left(\frac{500}{z} [\left| R_1 \right| + \left| R_2 \right| + \left| R_3 \right| + \left| R_4 \right|]\right). \]

**Exercise 3.** The closed-loop I/O transfer function is
\[ W(s) = \frac{K_d P(s)}{1 + K_d K_r P(s)} = \frac{K_d}{s + 1 + K_d K_r} \]

The steady state forced response to the input \( v(t) = 1 - t = v_1(t) - v_2(t) \) with \( v_1(t) = 1 \) and \( v_2(t) = t \)
\[ y_{ss}(t) = y_{ss,1}(t) - y_{ss,2}(t) = W(0) - (W(0)t + \frac{dW}{ds}|_{s=0}) = -W(0)t + (W(0) - \frac{dW}{ds}|_{s=0}) \]

We must require that \( y_{ss}(t) = 2t + 1 \) which implies
\[ -W(0) = 2, \ W(0) - \frac{dW}{ds}|_{s=0} = 1 \]

i.e.
\[ \frac{K_d}{1 + K_d K_r} = -2, \quad \frac{K_d}{(1 + K_d K_r)^2} = 3 \]

Moreover, for the existence of steady state regime we must require that the closed-loop is asymptotically stable, i.e. the closed-loop poles are in \( \mathbb{C}^- \):
\[ 1 + K_d K_r > 0 \]

From the first condition we obtain \( K_d = 4/3, K_r = -5/4 \) which however do not satisfy the second condition since \( 1 + K_d K_r = 1 - 20/12 < 0 \). We conclude that there are no values of \( K_r \) and \( K_d \) for which we have the desired steady state output response.