Exercise 1 By defining \( \bar{P}(s) = \frac{1}{s}(P(s) - 2) = -2\frac{s-5}{s(s+10)} \) and \( L(s) = G(s)\bar{P}(s) \), one has
\[
y(s) = W(s)v(s) + W_d(d(s) \quad \text{with} \quad W(s) = \frac{L(s)}{1+L(s)}, \quad W_d(s) = \frac{1}{1+L(s)}.
\]

Let us write \( G(s) = G_2(s)G_1(s) \) so that \( G_1(s) \) is designed for fulfilling steady-state specifications (i.e., \( (ii) \)) whereas \( G_2(s) \) will be later set for stability and transient performances (i.e., \( (i) \) and \( (iii) \)).

(ii) Since the input-to-error transfer function \( W_e(s) = \frac{1}{1+L(s)} \) and recalling that the steady state response to \( v(t) = t \) is given \( e_1(t) = W_e(0)t + \frac{dW_e}{ds}(0) \), for the requirement to be satisfied one needs \( W_e(0) = 0 \) and \( \left| \frac{dW_e}{ds} \right|_{s=0} \leq 0.2 \).

In this case, because an integrator is already located before the entering point of the disturbance, one has \( W_e(0) = 0 \) so that for \( (ii) \) to be solved one sets \( G_1(s) = k_1 \) with \( k_1 \in \mathbb{R} \) such that
\[
\left| \frac{W_e(s)}{s} \right|_{s=0} \leq 0.2 \implies k_1 \geq 5.
\]

Thus, one can fix \( k_1 = 5 \) while guaranteeing, for \( (ii) \) to be fulfilled by the closed-loop system, that \( G_2(0) \geq 1 \).

(iii) For assigning \( \omega_t^* = 2 \text{rad/sec} \) and \( m_\phi^* \geq 50^\circ \) let us first draw the Bode plots of
\[
L_1(s) = G_1(s)\bar{P}(s) = -10\frac{s-5}{s(s+10)} = \frac{5}{s(1 + \frac{5}{s})}
\]
which are reported in Figure 1. As \( \omega_t^* = 2 \text{rad/sec} \) is the desired crossover frequency, we notice that
\[
|L_1(j\omega_t^*)|_{dB} = 8.433 \quad \angle L_1(j\omega_t^*) = -123.1113.
\]

Accordingly, \( G_2(s) \) needs to be chosen in such a way that
\[
|G_2(j\omega_t^*)|_{dB} + |L_1(j\omega_t^*)|_{dB} = 0 \quad (2)
\]
\[
180^\circ + \angle L_1(j\omega_t^*) + \angle G_2(j\omega_t^*) \geq 50^\circ \quad (3)
\]
with the further requirement \( |G_2(0)| \geq 1 \) to preserve \( (ii) \).

It is a matter of computations to verify that, with no need of further actions, \( (3) \) is already satisfied as \( 180^\circ + \angle L_1(j\omega_t^*) = 56.8^\circ \). Accordingly, one can satisfy the specification by assigning the cross-over frequency to \( \omega_t^* = 2 \text{rad/sec} \). Thus, \( G_2(s) \) needs to be designed so to decrease the magnitude at \( \omega_t^* = 2 \text{rad/sec} \) without possibly affecting the phase in the corresponding neighborhood. By noticing that a simple proportional action is not
compatible with the requirement $|G_2(0)| \geq 1$, then an attenuating action of the form

$$G_2(s) = k_i \left( \frac{1 + \frac{\tau_i}{m_i} s}{1 + \tau_i s} \right)^{n_i}$$

is needed in such a way to guarantee

$$|G_2(j\omega_n^*)|_{dB} = -8.433, \quad \angle G_2(j\omega_n^*) \geq -6.8^o, \quad |k_i|_{dB} > 0.$$ 

As at $\omega_n^*$ must be affected as least as possible by the feedback action one has $k_i > 0$. Denoting $\omega_n = \omega_n^* \tau_i$, it is evident that for decreasing the phase contribution at $\omega_n^* = 2 \text{ rad/s}$ as much as possible, one needs $\omega_n$ to act at high frequency. Thus, we set $m_i = 3$, $\omega_n = 100$ and $n_i = 1$ so that

$$|G_2(j\omega_n^*)|_{dB} = |k_i|_{dB} - 9.5390, \quad \angle G_2(j\omega_n^*) = -1.1454^o.$$ 

Finally, the gain is set in such a way that $|k_i|_{dB} - 9.5390 = -8.433$ so resulting in $|k_i|_{dB} = 1.106 > 0$ which is indeed compatible with specification (ii) as $k_i = 1.1358 > 1$.

Figure 2 depicts the Bode plots of the open loop transfer function

$$L(s) = G_2(s)G_1(s)\bar{P}(s) = 11.358 \frac{1 + \frac{50}{2}s}{1 + 50s} \frac{s - 5}{s(s + 10)}. \quad (4)$$
(ii) The open loop system $L(s) = G_2(s)G_1(s)\bar{P}(s)$ possesses no poles with positive real part and one pole in zero with multiplicity one. Thus, the feedback system is asymptotically stable if and only if the number of counter-clockwise tours around $-1 + j0$ on behalf of the extended Nyquist plot of $L(j\omega)$ is 0. As Figure 3 suggests, the feedback system is asymptotically stable.

Exercise 2 (a) As the root locus of $P(s)$ describes the location of the poles of the feedback system under static feedback $G(s) = k$ (i.e., of the transfer function $W(s) = \frac{kP(s)}{1+kP(s)}$) the root locus of $P(s)$ is equivalent to the one of

$$\tilde{P}(s) = \frac{1}{(s-1)(s+2)}$$

deduced when neglecting the proportional term 3 and relabeling $\tilde{k} = 3k$.

Denoting by $n$ and $m$ respectively the number of poles and zeros, the relative degree is $r = n - m = 2$, the positive and negative locus possess respectively two branches. Moreover, the positive locus exhibits two vertical asymptotes centered at $s_0 = -1$.

Defining $p(s, \tilde{k}) = (s-1)(s+2) + \tilde{k}$ as the polynomial of the closed-loop poles, singularities
(s^*, \tilde{k}^*) \in \mathbb{C} \times \mathbb{R}$ are given by the solution of the coupled equations

$$p(s, \tilde{k}) = 0, \quad \frac{\partial}{\partial s} p(s, \tilde{k}) = 0$$

given by $s^* = -\frac{1}{2}$ and $\tilde{k}^* = 2$ (and thus $k = \frac{2}{3}$). Thus, the positive locus possesses a singularity of order two at $s^* = -\frac{1}{2}$ corresponding to $k = \frac{2}{3}$.

The positive and negative locus of $P(s)$ are thus reported in Figures 4 and 5.

(b) For assigning all poles with damping $\geq .7$, it is enough to assign them real. Moreover, as the root locus suggests, a static feedback is not enough for assigning all poles with real part $\geq 1$ as the center of the asymptotes is at $s_0 = -\frac{1}{2}$. Thus, a feedback $G(s)$ with dimension at least one is needed so to move the center of the asymptotes beyond $s = -1$. To preserve the relative degree and simplify the design, let us design a feedback of the form

$$G(s) = k_1 \frac{s + 2}{s + p}$$

with $k_1, p \in \mathbb{R}$ also generating uncontrollability of the mode associated to the eigenvalue $-2$. 

Figure 3: Nyquist plot of (4)
Figure 4: Positive root locus of $P(s) = \frac{3}{(s-1)(s+2)}$

At this point $p > 0$ needs to be set in such a way that

$$s'_0 = \frac{1-p}{2} \leq -1 \implies p > 3.$$  

For completing the design, it is enough to assign $p$ so to generate a singularity of order 2 at some $s^* \in \mathbb{R}$ and $s^* \leq -1$. It is a matter of computations to verify that such a singularity is unavoidably located in correspondence of the center of the asymptotes $s'_0$. Hence, one can set $p = 5$ in such a way that the closed-loop poles are located at $-2$ corresponding to $k_1 = 3$. Thus, the closed-loop transfer function is given by

$$W(s) = \frac{9}{(s + 2)^2}.$$  \hfill (5)

(c) As the disturbance is affecting the output, the output-disturbance transfer function is given by

$$W_d(s) = \frac{1}{1 + G(s)P(s)} = \frac{(s + 5)(s - 1)}{(s + 2)^2}.$$  

Accordingly, as $d(t) = t$, the steady state response is given by

$$y_{d,ss}(t) = \nabla_s W_d(0) + W_d(0)t$$  

with

$$W_d(0) = -\frac{5}{4}, \quad \nabla_s W_d(0) = -\frac{1}{4}.$$  

Exercise 3.
Figure 5: Negative root locus of $P(s) = \frac{3}{(s-1)(s+2)}$.

(i) The forced response to the input $u(t) = \text{cost}$ can be computed as

$$y_f(t) = \mathcal{L}^{-1}(Y(s))[t], \quad Y(s) = P(s)U(s) \quad U(s) = \mathcal{L}(u(t))[s]$$

with $\mathcal{L}$ and $\mathcal{L}^{-1}$ being the Laplace and inverse Laplace transforms.

As the system is in canonical controllable form, the transfer function $P(s) = C(sI - A)^{-1}B$ is given by

$$P(s) = \frac{1}{(s + a)^2}.$$ 

Accordingly, as $U(s) = \mathcal{L}(\text{cost})[s] = \frac{s}{s^2 + 1}$ one has

$$Y(s) = \frac{s}{(s + a)^2(s^2 + 1)} = \frac{R_{11}}{s + a} + \frac{R_{12}}{(s + a)^2} + \frac{As + B}{s^2 + 1}$$

with

$$R_{11} = \lim_{s \to -a} \nabla_s Y(s)(s + a)^2 = \frac{1 - a^2}{(1 + a^2)^2}$$

$$R_{12} = \lim_{s \to -a} Y(s)(s + a)^2 = -\frac{a}{1 + a^2}$$

$$A = \frac{a^2 - 1}{a^4 + 2a^2 + 1}$$

$$B = \frac{2a}{a^4 + 2a^2 + 1}.$$
Accordingly, by exploiting linearity of the Laplace operator one gets
\[
y_f(t) = R_{11} \mathcal{L}^{-1}\left(\frac{1}{s + a}\right)[t] + R_{12} \mathcal{L}^{-1}\left(\frac{1}{(s + a)^2}\right)[t] + A \mathcal{L}^{-1}\left(\frac{s}{s^2 + 1}\right)[t] + B \mathcal{L}^{-1}\left(\frac{1}{s^2 + 1}\right)[t] \\
= (R_{11} + tR_{12})e^{-at} + A \cos t + B \sin t.
\]

(ii) As the system only possesses one eigenvalue at \(s = -a\) with multiplicity 2, the output steady-state response only if the system is asymptotically stable that is if \(a > 0\). Accordingly, it can be easily deduced from the forced response by neglecting the terms whose effect vanish in time so getting
\[
y_{ss}(t) = A \cos t + B \sin t
\]
which can be rewritten as
\[
y_{ss}(t) = M \cos (t + \varphi)
\]
with \(M = |P(j)| = \frac{1}{a^2 - 1}, \varphi = -\angle(j + a)^2\).