Globally stable nonlinear flight control system


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Abstract: The use of linear, constant feedback control in automatic flight control systems for aircraft inevitably gives rise to formidable design difficulties when attempting to satisfy the conflicting requirements for faithful tracking of a pilot's manoeuvre commands and maintaining the aircraft's trimmed attitude in the presence of atmospheric turbulence. Two nonlinear control policies, VICTOR and ZOC, are shown to provide superior performance and, when used simultaneously in the same flight control system, to assure global stability. Such stability obviates the need to provide adaptive control or gain-scheduling schemes to satisfy the flying quality requirements over the entire flight envelope of an aircraft. The potential performance of the nonlinear control is demonstrated by means of some results obtained from a digital simulation of a pitch-rate manoeuvre-demand system for a typical medium jet transport aircraft.

List of symbols

- $c, c', c'', c'''$ = output variables
- $g$ = dynamic gain of the VICTOR scheme
- $h$ = non-negative number
- $k$ = a gain
- $k_1, k_2$ = lower and upper gain limits, respectively
- $k_x$ = a design constant
- $m$ = a system variable (see eqn. 28)
- $n$ = sensor noise
- $q$ = pitch rate of the aircraft
- $r$ = reference command variable
- $u$ = external disturbance to the control system
- $v$ = a system variable (see eqn. 13)
- $x$ = a system variable (see Fig. 5)
- $y$ = a system variable (see Fig. 3)
- $y'_1, y'_1$ = a system variable and its filtered value, respectively
- $z$ = a system variable (see eqn. 23)
- $\beta, \gamma, \delta$ = weighting factors
- $\eta$ = elevator deflection
- $\lambda_e$ = inverse time constant of the actuator
- $\sigma, \mu$ = design constants
- $\xi$ = damping ratio
- $A, B, C$ = system matrices
- $E$ = system performance measure, a scalar
- $F(s), G(s), G_1(s), G_A(s), H(s)$ = system transfer functions
- $L$ = a parameter used in the nonlinear stability criterion
- $N$ = a constraint, used in the performance measure
- $Q(\omega)$ = quadrature component of transfer function, $kG(j\omega)$
- $R(\omega)$ = reference component of transfer function, $kG(j\omega)$
- $T_v$ = a variable time constant
- $x$ = state vector

1 Introduction

For almost fifty years it has been customary in the design of an automatic flight control system (AFCS) for aircraft to regard the mathematical representation of the aircraft dynamics and its associated control law as comprising a linear feedback system [1—10]. The change of the stability derivatives with flight condition, the coupling of flight terms with high angles of attack, or, owing to rolling manoeuvres, the change of moments of inertia with time, and the use of a wind-axes system for the equations representing the aircraft motion have all been acknowledged [11—16] as having pronounced effects on the resulting flight response. For analytical convenience, however, these factors have usually been regarded as negligible second-order effects which, if they proved to be particularly troublesome in any specific case, could be dealt with by fitting auxiliary control systems. Such linear AFCS have been designed chiefly by using well established conventional methods employing either frequency-response or root-locus diagrams.

Although considerable effort has been directed since 1960 to the use of state-variable methods, few of the results obtained have proved to be acceptable from a viewpoint of flight operation. It has been shown, in Reference 17 for example, that for any dynamic system the number of outputs that can be controlled is no more than the number of control inputs available. Nor has it been possible by using linear state-variable feedback (LSVF) to position more of the closed-loop poles of the system than the maximum value of either the number of state variables of the system which can be sensed or the order of the reference vector of the closed-loop system [18]. Even were it possible to locate all the poles and zeros of a system, and even to control every output variable, the technique of using LSVF can be strongly criticised in engineering terms because it avoids the use of compensation elements (which can be cheaply synthesised with electronic components or digital algorithms) and uses instead transducers to measure the state variables. By their very functions, such transducers must be relatively expensive, and they also have significant nonlinearities and parasitic effects. Despite such obvious disadvantages, LSVF, obtained as a result of solving the linear quadratic problem (LQP) of modern control theory [19], has been widely applied, partly owing to the relative ease with which a numerical solution can be obtained, with stability of the closed-loop system assured (if the problem has been properly posed) by the feedback control law which such a solution provides.

From 1977, a series of new concepts involving nonlinear control policies was proposed by Gill [20—23]. To a considerable extent, these avoid the disadvantages of the LSVF methods. Gill's work, predominantly empirical and experimental in nature, has established by means of extensive simulation...
studies the effectiveness of such policies for AFCS application, but it has left open the question of whether such policies could assure global closed-loop stability.

In this paper, an heuristic proof is given that for one particular example of an AFCS, a pitch-rate maneouvre-demand system, global stability is guaranteed when two of the new methods proposed by Gill are employed simultaneously.

2 Limitations of linear AFCS

Any linear control law, however arrived at, can only provide the required closed-loop performance, expressed in terms of, say, the transient and steady-state responses, at the expense of permitting some unsatisfactory features in the response to disturbances to exist. Whether such disturbances are extraneous to the aircraft, such as can occur from its encountering atmospheric turbulence, or are introduced within the AFCS itself, arising from sensor noise for example, the design of any linear control must be such that, if the AFCS is arranged particularly to minimise the effects of such unwanted inputs, the desired dynamic closed-loop performance is unavoidably impaired. Even when there may be accepted some compromise which results in acceptable closed-loop responses, with some alleviation of the effects of disturbances, it can only be achieved within an extremely restricted region of the flight envelope. Consequently, when an aircraft is to move through the extreme regions of its flight envelope for some flight mission, recourse to gain-scheduling or self-adaptive schemes is frequently taken in an attempt to retain some measure of the compromise solution at every flight condition to be encountered.

![Diagram of linear time-invariant control system subject to disturbance and noise](image)

To illustrate this conflict of performance requirements, consider the following. For the system represented in Fig. 1, let the response of the controlled variable to some command input r be \( c' \); let the response of the same variable to some disturbance u be \( c'' \); and let the response to sensor noise n be \( c''' \). Then

\[
\frac{C'(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)} \tag{1}
\]

\[
\frac{C''(s)}{U(s)} = \frac{1}{1 + G(s)H(s)} \tag{2}
\]

\[
\frac{C'''(s)}{N(s)} = \frac{-G(s)H(s)}{1 + G(s)H(s)} \tag{3}
\]

and

\[
C(s) = C'(s) + C''(s) + C'''(s) \tag{4}
\]

If it is required that the output response \( c' \) should be negligibly small, then \( \{1 + G(s)H(s)\}^{-1} \) must be large, which can be achieved by arranging that

\[
G(s)H(s) \gg 1.0 \tag{5}
\]

In that event,

\[
C'(s) = \frac{R(s)}{H(s)} \tag{6}
\]

However, if it is required that for command input tracking \( c' \) should be identically equal to r, then, from eqn. 6, \( H(s) \) must be unity. Consequently, from expr. 5, \( G(s) \) must be very much greater than unity. But \( G(s) \) is influenced chiefly by the aircraft dynamics, which cannot be greater than unity over the entire spectrum. Moreover, to counter the effect of any disturbance u, if the inequality given in expr. 5 holds good, the measurement noise will appear at the output entirely undiminished, i.e.

\[
c''' = -n \tag{7}
\]

It is these contradictory results which constitute the performance conflict for linear AFCS.

Several methods [24—26] based on linear control theory have been proposed to mitigate the effects of the conflict, but they are generally unsuccessful in practical applications. The most common method is to insert in series in the forward path a compensation element (such as a phase advance filter) designed so that, over the frequency range of concern, the modulus of the frequency-response function associated with the modified loop transmittance, \( G_c(j\omega)G(j\omega)H(j\omega) \), is always greater than unity, without the associated phase characteristic having an adverse influence on the closed-loop stability. With wide changes in the parameters of \( G(s) \), such as often occurs over the flight envelope of any aircraft, it becomes impossible to design a fixed compensation element to fulful its purpose for every condition.

![Diagram of linear system with feedforward added](image)

Generally more successful in its application is the use of minor loop feedback compensation, since in practice it provides a more satisfactory solution over a wide range of variation of aircraft dynamics [27]. Yet difficulties are often experienced in providing a sufficient gain-bandwidth product in the minor loop to achieve the requirements for the design of the overall system. In addition, the complex function which results from the use of such a procedure may be either physically unrealisable or exceedingly complicated. It has been reported [28] that some benefit can be obtained by imposing a change on the structure of the control system to resolve the performance conflict. How such a structural change is imposed is indicated in Fig. 2, from which it can be seen that the system of Fig. 1 has been altered by introducing a feedforward path via the element F. It should be noted that here the aircraft dynamics, \( G_A(s) \), have been separated from the forward transfer function, \( G(s) \); \( G_A(s) \) represents the preamplifier, the actuator and any series compensation element which may also be employed. Thus, from
Above, are strong, and a suitable choice of $F(s)$ can be made for only a small restricted group of problems. When, too, $F(s)$ is measured to provide the necessary feedback, there will be associated with the sensors inevitable time lags, which then appear as derivative terms in eqn. 12, permitting such freedom to achieve the tracking requirement first achieving closed-loop stability and disturbance-rejection performance measure. The constants $a$ and $b$ depend on the parameters of the control loop as functions of on-line measurements $E$ and $N$, defined in eqns. 14 and 15) of particular aspects of system performance. How VICTOR operates may be explained with reference to the pitch-rate system represented by the block diagram of Fig. 3. First, assume that no noise is present and that, consequently, there is no need to include the noise filter. Next, suppose that the proportional-plus-integral element, $[1 + G_1(s)]$, is replaced by elements providing the relationship

$$v(p) = y(p) + x(p) = \left[1 + \frac{g}{p + (og - \mu)}\right]y_1(p) \quad (13)^\dagger$$

where

$$p \triangleq \frac{d}{dt}$$

$g$ is the dynamic gain of the VICTOR scheme$^5$; it is usually constrained to lie within some specified range (e.g. $0 \leq g \leq 1.0$) in correspondence with the instantaneous value of some performance measure. The constants $\sigma$ and $\mu$ depend on the particular flight-control mode being dealt with. Suppose that the performance measure $E$ of eqn. 14 is used$^6$, i.e.

$$E = \max \{\gamma |y_1(t)|, \beta |\dot{y}_1(t)|\} \quad (14)$$

and is subject to a constraint on the rate of change of the actuator $N$, i.e.

$$N = \frac{\beta \lambda_c}{(\lambda_c + p)} |\dot{y}_1(t)| \quad (15)$$

$^\dagger$Note the use of the operator $p = d/dt$. From this point in the paper, the operator $p$ is used exclusively so that the nonlinear elements always act on signals which are functions of time.

$^5$In Gill's work $g$ is referred to as the 'gearing'.

$^6$In practice the signals $y_1$ and $\dot{y}_1$ are not used directly but are obtained after being processed through first-order linear filters to become $y_1'$ and $\dot{y}_1'$.
\( \beta, \gamma \) and \( \delta \) are weighting factors which have to be chosen by the designer. The dynamic gain \( g \) of the scheme is defined as the difference between these two positive measures, i.e.

\[
g = E - N
\]  

\( \text{Fig. 4} \) Generation of dynamic gain of VICTOR scheme

\[
\begin{align*}
x(p) & = \frac{0.1}{p} \\
y'(p) & = 1 + 0.1 \\
g & = E - N
\end{align*}
\]

\( \text{Fig. 5} \) An implementation of VICTOR

How the law can be implemented is shown in Fig. 4. With \( g \) available, eqn. 13 can be synthesised in the manner indicated in Fig. 5. For example, if \( a = 10 \mu \) and \( g \) lies in the range \( 0.1 \leq g < 1.0 \), then, as \( g \) varies throughout its range, the transfer operator \( x(p)/y_1(p) \) is changed from

\[
\frac{x(p)}{y'_1(p)} = \frac{0.1}{p}
\]

to

\[
\frac{x(p)}{y'_1(p)} = \frac{1}{p + 9\mu}
\]

Therefore

\[
\frac{v(p)}{y'_1(p)_{g=0.1}} = \frac{p + 0.1}{p} = 1 + 0.1
\]

and

\[
\frac{v(p)}{y'_1(p)_{g=1.0}} = \frac{p + 9\mu + 1}{p + 9\mu} \approx 1.0 \quad 9\mu \gg 1
\]

When \( |y'_1| > |y'_1| \), \( g \) tends to decrease to its lowest value, which causes \( y'_1 \) to decrease. If \( |y'_1| \) is too large, \( g \) increases, thereby reducing \( y'_1 \). In practice, \( g \) operates in association with \( G_n \), which is chosen to ensure that the performance of the closed-loop system is acceptable for either extreme value of \( g \). The dynamic gain \( g \) of the VICTOR scheme therefore changes only when the system is responding to some command input, when it is perturbed by measurement noise, or when it is subjected to external disturbances. Consequently, the performance conflict outlined in Section 2 is largely circumvented.

The effectiveness of the VICTOR scheme can be inferred from the responses illustrated in Fig. 6. These responses were obtained from a digital simulation of an SAS, of the type shown in Fig. 3, for a transport aircraft whose equations of motion are detailed in the Appendix. The aircraft is considered to be subjected to an up-gust (an external disturbance) and also to measurement noise. The responses shown represent three cases:

(i) when the VICTOR gain \( g \) was fixed at its lowest value \( 0.1 \)

(ii) when \( g \) was fixed at its upper limit \( 1.0 \)

(iii) when the VICTOR system was operating so that \( g \) was changing dynamically with system signals. It is emphasised that a second system, ZOC, was operating in conjunction with VICTOR in this case.

It should be evident from Fig. 6a that when the fixed gain is low the resulting pitch-rate response to the very moderate up-gust is persistent and of relatively large amplitude. The elevator activity \( \gamma \) is very high, owing to noise, although the levels of deflection are not great. However, for large \( g \) (and hence large loop gain) low-amplitude pitch rate is achieved with minimum noise effects, and yet the dynamic response is very oscillatory and thereby unacceptable for long flights. The use of VICTOR and ZOC achieves the required compromise: well controlled pitch-rate response with acceptable damping and low control surface activity, even in the presence of noise. Note, however, that the dynamic gain alters rapidly throughout the response. Why ZOC is required to be used with VICTOR is explained in a later Section, but first an account of ZOC is given.

5 Zero overshoot control (ZOC)

It would be more accurate to regard the second nonlinear policy as being piecewise linear. It depends for its operation on predicting the likely occurrence of an overshoot in the system error when that signal is being restored to its equilibrium value. If such an overshoot is likely to occur, the policy operates to prevent it. The control policy is therefore referred to as zero overshoot control (ZOC). The basis upon which the ZOC system automatically predicts that an overshoot is likely to occur is the too-rapid decrease of the error signal.

Consider the curves in Fig. 7, for example. They represent the error signals of two simple second-order linear time-variant systems. It is evident that when the damping ratio of such a system is unity \((\xi = 1.0)\) there is no overshoot; when \( \xi = 0.2 \) an overshoot occurs. The ZOC system must detect the likelihood of occurrence of this overshoot and modify the parameters of its closed-loop system to achieve a dynamic response like that represented by the broken line in Fig. 7. Overshoots may occur, of course, in aperiodic systems [29], and, although the modes of some higher-order systems may be lightly damped, the overall response of a high-order system may exhibit no overshoot if the other, dominant, modes are aperiodic [30]. Thus it should be appreciated that the ZOC system is best implemented to determine overshoots experimentally. Any convenient technique may be used, although some have disadvantages which preclude their adoption for practical applications. The particular ZOC scheme discussed in the remainder of this paper is one that was developed by Gill and his co-workers [23]. The single noise filter of Fig. 3 is replaced by two cascaded noise filters, and a phase-advance network has been inserted into the loop; see Fig. 8. The gain \( a \) of the phase-advance element is applied only when the error signal \( y'_1 \) is decreasing.
too rapidly, i.e. when
\[ \text{sgn}(k_x y'_1 + y'_1) \neq \text{sgn}(y'_1) \] (21)

\[ k_x \] is a constant which has to be determined by the designer. The overdamped response shown in Fig. 7 was achieved by using a ZOC system based on expr. 21.

6 Pitch-rate manoeuvre demand system

The principal advantage of employing both the VICTOR and ZOC schemes is the marked improvement in the dynamic performance of the closed-loop system which can be achieved without any great difficulty in maintaining the required stability properties.

To demonstrate the very stable nature of such schemes, a particular example of a pitch-rate manoeuvre demand system designed to operate on a BAC 1-11 medium-size twin-jet transport aircraft is considered. The block diagram of this AFCS is presented in Fig. 9. Note how the dynamic

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**Fig. 6** Responses of baseline SAS and SAS plus VICTOR

**Fig. 7** Representative error signals of a simple second-order system

**Fig. 8** ZOC scheme
gain \( g \) of the VICTOR scheme influences both the integral path and the variable filter. The ZOC scheme in Fig. 9 influences the washout network in the error channel so that, when ZOC operates, the washout circuit and the noise filter provide some amount of phase advance. However, when \( a \) is set to zero, the washout effect is annulled and only the simple phase lag of the noise filter is effective.

To use Popov's criterion [31]. This approach requires that the closed-loop system be regarded as comprising a linear element in the forward path and a nonlinear element in the feedback path. To obtain the system equation most easily in this form, it is best to rearrange the block diagram of Fig. 11 into the alternative form of Fig. 12, from which the forward transfer operator \( G(p) \) and the nonlinear feedback element \( \phi(g) \) may be determined by inspection:

\[
G_{ZOC}(p) = \frac{0.72(1 + ap)(1 + 1.57p)}{(1 + 0.02p)^2(1 + 0.14p)(p^2 + 1.6p + 1.41)}
\]

(22)

and

\[
z = -\phi(g)m
\]

(23)
where
\[
\phi(g) = \frac{g(6 + 14g^2) \left( 1 + \frac{6}{6 + 14g^2} \right)}{(1 + T_g p)(1 + p)}
\]  \hspace{1cm} (24)

Note that eqn. 22 depends on the state of the ZOC system: if, momentarily, ZOC is not operative, then \( G \) becomes, for a moment,
\[
G_{ZOC}(p) = \frac{0.72(1 + 1.5p)}{(1 + 0.02p)^2(1 + 0.14p)(p^2 + 1.64 + 1.41)}
\]  \hspace{1cm} (25)

Note, too, from eqn. 24 and from Fig. 11 that the gain of the nonlinear element \( \phi \) goes from 6 to 20 as \( g \) changes from 0.0 to 1.0. The time constant of the numerator of \( \phi \) also decreases from 1.0 to 0.3 s as \( g \) increases. Thus both the gain and the time constant of the nonlinear element are state dependent, for \( g \) depends on the aircraft state variables for its instantaneous value.

7 Popov's criterion of stability

The differential equation of the system represented in Fig. 12 can be expressed as
\[
\dot{x} = Ax + Bz
\]  \hspace{1cm} (26)

where
\[
z = \phi(x)
\]  \hspace{1cm} (27)

and
\[
m = Cx
\]  \hspace{1cm} (28)

Popov has shown [31] that for the nonlinear system represented by eqns. 26–28 the equilibrium solution is asymptotically stable in the large if the following conditions are met:

(i) The nonlinear feedback element \( \phi(x) \) is time invariant and, for all \( x \), satisfies
\[
k_1 < \phi(x) < k_2
\]  \hspace{1cm} (29)

where \( k_1 \) and \( k_2 \) are lower and upper bounds, respectively.

(ii) The transfer operator \( F(p) \), given by
\[
F(p) = \frac{k_2 G(p) + 1}{k_1 G(p) + 1}
\]  \hspace{1cm} (30)

is positive real, i.e. it has the following properties:

(a) It possesses no poles with positive real parts.

(b) If it possesses poles with zero real parts, they are simple and have positive, real residues.

(c) It is real for real values of \( p \).

(d) \( \text{Re} \{F(j\omega)\} \geq 0 \) must hold for all real \( \omega \).

This theorem is usually referred to as the circle criterion.

8 Asymptotic stability of a pitch-rate manoeuvre demand system

The transfer operator \( G(p) \) of the linear element of the system represented by Fig. 12 depends on the state of the ZOC system. It has been shown earlier how, in this appli-
cation, ZOC is essentially an instantaneous switching function which switches in/out some phase advance but does not affect the gain of the forward path. The polar diagrams for $G_{ZOC}(j\omega)$ and $G_{ZOC_1}(j\omega)$ are shown in Fig. 13: when ZOC ceases to operate, the frequency-response locus switches instantaneously from curve $b$ to curve $a$.

Because the dynamic gain $g$ of the VICTOR scheme lies in the range $0.0$ to $1.0$, the upper limit of the nonlinear gain is $20.0$, and its lower limit is zero. Hence the bounds $k_1$ and $k_2$ of expr. 29 are 0.0 and 20.0, respectively. The function $F(p)$ therefore becomes

$$F(p) = 20G(p) + 1$$  \hspace{1cm} (31)

From eqn. 24, it is evident that $\phi(p)$ is a function only of $x$ and does not depend explicitly on time. Furthermore, the Popov criterion does not require any solution of $\phi(p)$, merely that the maximum and minimum values of the gain of the nonlinear feedback element be bounded. Hence, to show that the pitch-rate control system is ASIL, it is both necessary and sufficient to show that $F(p)$ of eqn. 31 is a positive real function.

Whether ZOC operates or not, the poles of $F(p)$ are:

$$\begin{align*}
- 50.0 \\
- 50.0 \\
- 7.143 \\
- 0.81 + i0.87 \\
- 0.81 - i0.87
\end{align*}$$  \hspace{1cm} (32)

It is evident that no pole has a positive real part or a zero real part. Property (d) from condition (ii) in Section 7 is not sufficient, however, for positive realness of $F(p)$. The zeros of $r(p) + kq(p)$ must have negative real parts for $k_1 < k < k_2$. Since $k_1$ is zero, the poles of $r(p)$ are seen to be negative from inspection. The poles of $r(p) + kq(p)$ are quoted in eqn. 32. For any $p$, $F(p)$ is real. For example, if $p = 2$, say, then

$$F_{ZOC_0}(2) = 0.209$$  \hspace{1cm} (33)

and

$$F_{ZOC_1}(2) = 0.251$$  \hspace{1cm} (34)

The final check is to determine if $\text{Re} \{F(j\omega)\} > 0$ for all real $\omega$. From Fig. 14, which shows the polar diagram of $F(j\omega)$ for $0 < \omega < \infty$, it is obvious that, for $ZOC_1$, $\text{Re} \{F(j\omega)\} > 0$ for all $\omega > 0$ (curve $b$); for $ZOC_0$ (curve $a$) this does not hold.

The graphical interpretation which the circle criterion allows is the method by which the final step in the demonstration of the global stability of the system is taken. The condition

$$\Re \left( \frac{k_2 G(j\omega) + 1}{k_1 G(j\omega) + 1} \right) > 0$$  \hspace{1cm} (35)

for all $\omega > 0$, is checked on the polar diagram of $G(j\omega)$, Fig. 13. The segment of the real axis between $-1/k_1$ and $-1/k_2$ is the diameter of a circle in the complex plane. Eqn. 35 requires that the locus of $G(j\omega)$ must not lie in that region of the complex plane enclosed by the circle whose diameter is $(1/k_1 - 1/k_2)$ and whose centre is $-1/2 (1/k_1 + 1/k_2)$. Since $k_1$ is zero, the arc of the circle passing through $-1/k_2$ on the polar diagram is a straight line: the critical line marked on Figs. 13 and 14.

When ZOC is not operating momentarily, and curve $I$ obtains, $g$ will be at some value lower than its maximum and, consequently, the critical line will lie further to the left than shown in either Fig. 13 or 14. When the VICTOR system operates, the dynamic gain is increased and, therefore, the critical line is moved to the right; if no ZOC system were to operate, the critical line and the $G(j\omega)$ locus would intersect at some value lower than $g = 1.0$, and therefore instability would ensue. Before that occurs, however, in this flight control system the phase margin would be decreasing, causing the dynamic response of the closed-loop system to become oscillatory. ZOC operates by detecting such rapidity of change, which indicates likely overshoot. As the dynamic gain increases rapidly, therefore, ZOC is caused to operate and shift, by virtue of the added phase advance, the frequency-response locus to curve $B$, which means that even if the VICTOR system causes the nonlinear gain to reach its maximum value, $k_2$, the pitch-rate, manoeuvre-demand system remains stable, for curve $B$ never intersects the critical line.

Thus the ASIL property of the proposed flight control system has been demonstrated.

9 Some features of VICTOR and ZOC

Global stability could be obtained for the proposed system without using ZOC merely by reducing $k_2$. However, with too low a value of loop gain the pitch-rate system cannot provide the dynamic response required for satisfactory handling of the aircraft.

The dynamic gain of the VICTOR scheme has no direct influence on the stability properties other than affecting the instantaneous value of the nonlinear gain (and hence the location of the critical line) and also in causing the ZOC scheme to operate in the manner described earlier in the paper. Yet it has a direct and important effect upon the response of the closed-loop system. If the VICTOR scheme is arranged to secure some acceptable, low value of performance index, corresponding to the closed-loop system response, and were the ZOC scheme to fail, for example, the resulting oscillatory response (caused by the nonlinear gain being adjusted towards $k_2$) would increase the performance index. This increase would cause, in turn, a reduction in the dynamic gain, $g$, of the VICTOR scheme because it depends upon the error or the error rate whichever is the greater (see eqn. 14). Such a reduction in $g$ would restore the performance index to an acceptable value and would, at the same time, restore the system’s stability.

The rate at which $g$ can change is controlled by the phase lead term in eqn. 24. As $g$ increases, the time constant of the phase advance is reduced (see Fig. 11c); i.e. as the VICTOR system causes the critical line of Figs. 13 or 14 to

Fig. 14  Polar diagram for application of Popov criterion

- $a$  $G_{ZOC}$
- $b$  $G_{ZOC_1}$

$ZOC$ affects only the zeros of $F(p)$.  

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be moved to the right towards the locus $G(j\omega)$ or $F(j\omega)$ the speed at which the line approaches the locus is reduced by the reduction in the time constant of the phase advance network. Any incipient transient effects due to rapid changes in gain are minimised by the addition of a filter with a variable time constant (see Fig. 9). It is emphasised that these terms, ZOC, VICTOR and variable time constant filter, are arranged to work in harmony by means of the slaved relationship shown graphically in Fig. 11.

10 Bounded-input bounded-output stability

In the proof of ASIL, the presence of either noise or command signals was ignored. It remains essential, however, to determine whether or not the pitch-rate control system is bounded in its response to bounded inputs.

A number of results are available on the stability of nonlinear systems excited by external forces [32—34]. In particular, the off-axis circle criterion of Cho and Narendra [35] provides results which, for the scalar system being considered, its response to bounded inputs.

In the proof of ASIL, the presence of either noise or command signals was ignored. It remains essential, however, to determine where

where $i$ is an arbitrarily small positive number.

It is worth noting that this theorem does not require that the poles of $G(p)$ lie in the left-half plane. If there are any poles located there, for a particular aircraft, $h$ is taken to be zero. Note also that the theorem involves the derivative of the nonlinear function rather than the function itself.

By defining a parameter $L$ such that

$$L = \frac{k_2}{h}$$

(40)

expr. 38 may be re-expressed as

$$\text{Re} \left( \frac{k_2 G(j\omega)}{L + k_2 G(j\omega)} \right) + \frac{1}{L - 1} \geq 0 \quad \text{for } 0 \leq \omega < \infty$$

(41)

Setting

$$k_2 G(j\omega) = R(\omega) + jQ(\omega)$$

(42)

and replacing the inequality given in expr. 41 by an equation, namely

$$R(\omega) + \frac{1}{2} (L + 1)^2 + Q^2(\omega) = \frac{1}{2} (L - 1)^2$$

(43)

results in the definition of a family of circles having radii $(L - 1)/2$ and lying to the left of the line $R(\omega) = -1$. If the locus of $k_2 G(j\omega)$ in the $RQ$-plane, for any given value of $k_2$, is found to be tangent to a circle with radius $L$, say, then the value of that radius determines the value of $h$ which defines the limits of $d\phi(\xi)/d\xi$ required to guarantee absolute stability. The dependency of $\phi$ on $\xi$ need not be explicit, but $\phi$ depends explicitly upon $g$, which depends, in turn, on $y_1$ or $y_2$, whichever is the greater. From eqn. 24,

$$\phi = \frac{(6g + 14g^2) + p \left\{ \frac{36g + 84g^2}{6 + 14g^2} \right\}}{0.02p + \left\{ \frac{0.02}{(0.1 + 0.9g)} \right\} p + 1}$$

(44)

Although it is evident from eqn. 44 that $d\phi/dg$ will be a complex expression, it can be shown to be physically realisable: for any limiting value of $h$ it is possible, therefore, to determine the corresponding value of $g$. It can therefore be imagined that when the VICTOR and ZOC schemes are operating they generate a dynamic circle, i.e. one with a radius $L$ which changes with time. Consequently, it can be envisaged that this circle shrinks when it is required to preserve stability and that it expands when it is required to recover high performance.

The theorem of Naumov and Tsypkin is less restrictive than either the circle or off-axis circle criteria quoted earlier. Consequently, whenever the input $r(t)$ is known, or is chosen, to be bounded and the system has been designed to meet the requirements of either the circle or off-axis circle criteria, the dynamic response of the pitch-rate control system will be more stable than predicted.

11 Conclusions

The stability properties of a particular form of the VICTOR and ZOC techniques applied to a typical pitch-rate manoeuvre-demand system have been shown to be ASIL and BIBO stable. Thus absolute stability is achieved while the system provides at the same time a dynamic performance which meets the several conflicting requirements typical of this kind of application. Because of the self-stabilising properties
of the system, wide variations can occur in the plant parameters or the controller settings to meet the immediate performance requirements. The scheme lends itself to digital synthesis, and it appears to be successful even for large-order systems, provided only that the 'slave relationship' of the parameters of the control system is maintained. The exact form of these slave relationships has been arrived at experimentally, and a thorough investigation is starting shortly to determine a sound theoretical basis for their choice. No serious loss of performance has been encountered, however, in the experimental investigations when different functional relationships have been used, provided the slaving relationships were observed.

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14 Appendix: Aircraft dynamics

The equations of motion used in the study represented the longitudinal motion of a BAC 1—11, twin-jet, medium-size, commercial transport aircraft cruising at 0.6 Mach at 30 000 ft:

\[
\ddot{w} = -0.686w + 1.14q - 0.054\eta \\
\dot{q} = -0.82w - 0.236\dot{w} - 0.658\dot{q} - 1.14\eta
\]

where \(w\) represents the vertical velocity in metres per second, \(q\) represents the pitch rate in degrees per second and \(\eta\) represents the elevator deflection in degrees. It is easily shown that eqns. 45 and 46 may be written in state-variable form as

\[
\begin{bmatrix}
\dot{w} \\
\dot{q}
\end{bmatrix} =
\begin{bmatrix}
-0.686 & 1.14 \\
-0.658 & -0.954
\end{bmatrix}
\begin{bmatrix}
w \\
q
\end{bmatrix} +
\begin{bmatrix}
0.054 \\
-1.128
\end{bmatrix}\eta
\]

i.e.

\[
\dot{x} = Ax + B\eta
\]

where

\[
x' = [w \ q] \\
y \hat{=} q = [0 \ 1]x = Cx
\]

The transfer operator is

\[
\frac{q(p)}{\eta(p)} = C[pI - A]^{-1}B
\]

which can be shown to be

\[
\frac{q(p)}{\eta(p)} = \frac{-0.72(1 + 1.57p)}{(p^2 + 1.64p + 1.41)}
\]