Optimal Control

Lecture

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Schedule
Tuesday: 14:00-15:30 (A6)
Wednesday: 14:00-15:30 (A6)

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Grading
Project + oral exam

The exam must be concluded before the second part of Identification that will be held by Prof. Battilotti
Grading
Project+ oral exam

Example of project:
- Read a paper on an optimal control problem
- Study: background, motivations, model, optimal control, solution, results
- Simulations

You must give me, before the date of the exam:
- A .doc document
- A power point presentation
- Matlab simulation files

The exam must be concluded before the second part of Identification that will be held by Prof. Battilotti
Some projects studied in 2014-15

Application of Optimal Control to malaria: strategies and simulations

Performance compare between LQR and PID control of DC Motor

Optimal Low-Thrust LEO (low-Earth orbit) to GEO (geosynchronous-Earth orbit) Circular Orbit Transfer

Controllo ottimo di una turbina eolica a velocità variabile attraverso il metodo dell'inseguimento ottimo a regime permanente

Optimal Control in Dielectrophoresis

On the Design of P.I.D. Controllers Using Optimal Linear Regulator Theory

Rocket Railroad Car

.........
THESE SLIDES ARE NOT SUFFICIENT FOR THE EXAM: YOU MUST STUDY ON THE BOOKS

Part of the slides has been taken from the References indicated below
References


L. Evans, *An introduction to mathematical optimal control theory*, 1983


How, Jonathan, *Principles of optimal control*, Spring 2008. (MIT OpenCourseWare: Massachusetts Institute of Technology). License: Creative Commons BY-NC-SA.
Course outline

• Introduction to optimal control
• Nonlinear optimization
• Dynamic programming
• Calculus of variations
• Calculus of variations and optimal control
Course outline

- **Introduction to optimal control**
- Nonlinear optimization
- Dynamic programming
- Calculus of variations
- Calculus of variations and optimal control
First course on linear systems (free evolution, transition matrix, gramian matrix,...)
Notations

\[ x(t) \in \mathbb{R}^n \quad \text{State variable} \]

\[ u(t) \in \mathbb{R}^p \quad \text{Control variable} \]

\[ f : \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R} :\rightarrow \mathbb{R} \]

Function \( \mathcal{C}^2 \) (function with second derivative continuous a.e.)
Introduction

Optimal control is one particular branch of modern control that sets out to provide analytical designs of a special appealing type. The system that is the end result of an optimal design is supposed to be the best possible system of a particular type.

A cost index is introduced
Introduction

Linear optimal control is a special sort of optimal control:

✓ the plant that is controlled is assumed linear
✓ the controller is constrained to be linear

Linear controllers are achieved by working with quadratic cost indices
Introduction

Advantages of linear optimal control

✓ Linear optimal control may be applied to nonlinear systems
✓ Nearly all linear optimal control problems have computational solutions
✓ The computational procedures required for linear optimal design may often be carried over to nonlinear optimal problems
History

1696: THE BIRTH OF OPTIMAL CONTROL

Jan C. Willems
Department of Mathematics
University of Groningen

Proceedings of the 35th
Conference on Decision and Control
Kobe, Japan • December 1996
In 1696 Bernoulli posed the **Brachystochrone problem** to his contemporaries: “it seems to be the First problem which explicitly dealt with optimally controlling the path or the behaviour of a dynamical system”.
Motivations

Example 1 (Evans 1983)
Reproductive strategies in social insects
Let us consider the model describing how social insects (for example bees) interact:

\[ w(t) \] represents the number of workers at time \( t \)
\[ q(t) \] represents the number of queens
\[ u(t) \] represents the fraction of colony effort devoted to increasing work force
\[ T \] length of the season
Death rate

\[ \dot{w}(t) = -\mu w(t) + b s(t)u(t)w(t) \]

\[ w(0) = w_0 \]

Known rate at which each worker contributes to the bee economy

\[ \dot{q}(t) = -\nu q(t) + c(1-u(t))s(t)w(t) \]

\[ q(0) = q_0 \]

Evolution of the worker population

Evolution of the Population of queens

Constraint for the control: \[ 0 \leq u(t) \leq 1 \]
The bees goal is to find the control that maximizes the number of queens at time $T$:

$$J(u(t)) = q(T)$$

The solution is a \textit{bang-bang control}.
Motivations

Example 2 (Evans 1983...and everywhere!)

A moon lander

**Aim:** bring a spacecraft to a soft landing on the lunar surface, *using the least amount of fuel*

- $h(t)$ represents the **height** at time $t$
- $v(t)$ represents the **velocity** = $\dot{h}(t)$
- $m(t)$ represents the **mass** of spacecraft
- $u(t)$ represents **thrust** at time $t$

We assume $0 \leq u(t) \leq 1$
Consider the Newton’s law: \( m\ddot{h}(t) = -gm + u \)

\[
\dot{v}(t) = -g + \frac{u(t)}{m(t)} \\
\dot{h}(t) = v(t) \\
\dot{m}(t) = -ku(t) \quad h(t) \geq 0 \quad m(t) \geq 0
\]

We want to minimize the amount of fuel that is maximize the amount remaining once we have landed

where

\( \mathcal{I} \) is the first time

\[
h(\mathcal{I}) = 0 \quad v(\mathcal{I}) = 0
\]
Analysis of linear control systems

Essential components of a control system
✓ The plant
✓ One or more sensors
✓ The controller
Analysis of linear control systems

**Feedback:** the actual operation of the control system is compared to the desired operation and the input to the plant is adjusted on the basis of this comparison.

**Feedback control systems** are able to operate satisfactorily despite adverse conditions, such as disturbances and variations in plant properties.
Course outline

• Introduction to optimal control

• **Nonlinear optimization**

• Dynamic programming

• Calculus of variations

• Calculus of variations and optimal control
Definitions

Consider a function $f : \mathbb{R}^n \rightarrow \mathbb{R}$

And $D \subseteq \mathbb{R}^n$

$\| \cdot \|$ denotes the Euclidean norm

A point $x^* \in D$ is a local minimum of $f$ over $D \subseteq \mathbb{R}^n$

If $\exists \, \varepsilon > 0$ such that for all $x \in D$ satisfying $\| x - x^* \| < \varepsilon$

$\Rightarrow$

$f(x^*) \leq f(x)$
Definitions

Consider a function \( f : R^n \rightarrow R \)

And \( D \subseteq R^n \)

\( \| \cdot \| \) denotes the Euclidean norm

A point \( x^* \in D \) is a **strict local minimum** of \( f \) over \( D \subseteq R^n \)

If \( \exists \ \varepsilon > 0 \) such that for all \( x \in D \) satisfying \( |x - x^*| < \varepsilon \)

\[ f(x^*) < f(x), \ \forall x \neq x^* \]
Definitions

Consider a function $f : \mathbb{R}^n \rightarrow \mathbb{R}$

And $D \subseteq \mathbb{R}^n$

$\| \cdot \|$ denotes the Euclidean norm.

A point $x^* \in D$ is a **global minimum** of $f$ over $D \subseteq \mathbb{R}^n$

If

$\text{for all } x \in D$

$\Rightarrow$

$f(x^*) \leq f(x)$
Definitions

The notions of a **local/strict/global maximum** are defined similarly.

If a point is either a maximum or a minimum is called an **extremum**.
Unconstrained optimization - first order necessary conditions

All points $x$ sufficiently near $x^*$ in $\mathbb{R}^n$ are in $D$

Assume $f \in C^1$ and $x^*$ its local minimum. Let $d \in \mathbb{R}^n$ an arbitrary vector.

Being in the unconstrained case:

$x^* + \alpha d \in D \ \forall \alpha \in \mathbb{R} \text{ close enough to } 0$

Let’s consider:

$g(\alpha) := f(x^* + \alpha d)$

0 is a minimum of $g$
**Unconstrained optimization** - first order necessary conditions

First order Taylor expansion of $g$ around $\alpha = 0$

$$g(\alpha) = g(0) + g'(0)\alpha + o(\alpha), \quad \lim_{\alpha \to 0} \frac{o(\alpha)}{\alpha} = 0$$

$g'(0) = 0$

**Proof:** assume $g'(0) \neq 0$

$$\exists \varepsilon > 0 \text{ small enough so that }$$

$$\text{for } |\alpha| < \varepsilon \quad |o(\alpha)| < |g'(0)\alpha|$$

For these values of $\alpha$

$$g(\alpha) - g(0) < g'(0)\alpha + |g'(0)\alpha|$$
Unconstrained optimization - first order necessary conditions

If we restrict \( \alpha \) to have the opposite sign to \( g'(0) \)

\[
g(\alpha) - g(0) < g'(0)\alpha + |g'(0)\alpha|
\]

\[\Rightarrow g(\alpha) - g(0) < 0\]

contradiction.

\[
g'(\alpha) = \nabla f(x^* + \alpha d) \cdot d \quad \text{where} \quad \nabla f := \begin{pmatrix} f_{x_1} & \cdots & f_{x_n} \end{pmatrix}^T \quad \text{is the gradient of} \quad f
\]

\[\Rightarrow g'(0) = \nabla f(x^*) \cdot d = 0\]

\( d \) was arbitrary

\[\nabla f(x^*) = 0\]

First order necessary condition for optimality
Unconstrained optimization - first order necessary conditions

A point $x^*$ satisfying this condition is a **stationary point**
Unconstrained optimization - second order conditions

Assume \( f \in C^2 \) and \( x^* \) its local minimum. Let \( d \in \mathbb{R}^n \) an arbitrary vector.

**Second order Taylor** expansion of \( g \) around \( \alpha = 0 \)

\[
g(\alpha) = g(0) + g'(0)\alpha + \frac{1}{2} g''(0)\alpha^2 + o(\alpha^2),
\]

\[
\lim_{\alpha \to 0} \frac{o(\alpha^2)}{\alpha^2} = 0
\]

Since \( g'(0) = 0 \)

\[
g''(0) \geq 0
\]

**Proof**: suppose \( g''(0) < 0 \)

\[
\exists \varepsilon > 0 \text{ small enough so that }
\]

\[
\text{for } |\alpha| < \varepsilon \quad |o(\alpha^2)| < \frac{1}{2} |g''(0)|\alpha^2
\]
Unconstrained optimization - second order conditions

For these values of \( \alpha \)
\[
g(\alpha) - g(0) < 0
\]

contradiction

By differentiating both sides with respect to \( \alpha \)

\[
g'(\alpha) = \sum_{i=1}^{n} f_{x_i}(x^* + \alpha d)d_i
\]

\[
g''(\alpha) = \sum_{i,j=1}^{n} f_{x_ix_j}(x^* + \alpha d)d_i d_j
\]

\[
g''(0) = \sum_{i,j=1}^{n} f_{x_ix_j}(x^*)d_i d_j = d^T \nabla^2 f(x^*)d
\]

\[
\nabla^2 f = \begin{pmatrix}
  f_{x_1x_1} & \cdots & f_{x_1x_n} \\
  \vdots & \ddots & \vdots \\
  f_{x_nx_1} & \cdots & f_{x_nx_n}
\end{pmatrix}
\]

Hessian matrix
Unconstrained optimization - second order conditions

\[ \nabla^2 f(x^*) \geq 0 \]

Second order necessary condition for optimality

Remark:
The second order condition distinguishes minima from maxima:

At a **local maximum** the Hessian must be **negative semidefinite**

At a **local minimum** the Hessian must be **positive semidefinite**
Let $f \in C^2$ and $\nabla f(x^*) = 0$ $\nabla^2 f(x^*) > 0$

$x^*$ is a **strict local minimum** of $f$
A vector $d \in \mathbb{R}^n$ is a **feasible direction** at $x^*$ if

$$x^* + \alpha d \in D \text{ for small enough } \alpha > 0$$

If $D$ in not the entire $\mathbb{R}^n$ then $D$ is the **constraint set** over which $f$ is being minimized.
Global minimum

**Weierstrass Theorem**

Let $f$ be a *continuous function* and $D$ a *compact set*

there exist a *global minimum* of $f$ over $D$
Constrained optimization

Let $D \subset \mathbb{R}^n$, $f \in C^1$

Equality constraints $h(x) = 0$, $h : \mathbb{R}^p \to \mathbb{R}$, $h \in C^1$

Inequality constraints $g(x) \leq 0$, $g : \mathbb{R}^q \to \mathbb{R}$, $g \in C^1$

Regularity condition:

\[
\text{rank}\left\{ \left. \frac{\partial (h, g_a)}{\partial x} \right|_{x^*} \right\} = p + q_a 
\]

where $g_a$ are the active constrain of $g$ with dimension $q_a$

Lagrangian function $L(x, \lambda_0, \lambda, \mu) = \lambda_0 f(x) + \lambda^T h(x) + \eta^T g(x)$

If $\lambda_0 \neq 0$ the stationary point $x^*$ is called normal and we can assume $\lambda_0 = 1$. 

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Constrained optimization

From now on $\lambda_0 = 1$ and therefore the Lagrangian is

$$L(x, \lambda, \mu) = f(x) + \lambda^T h(x) + \eta^T g(x)$$

If there are only equality constraints the $\lambda_i$ are called Lagrange multipliers

The inequality multipliers are called Kuhn – Tucker multipliers
Constrained optimization

First order necessary conditions for constrained optimality:

Let \( x^* \in D \) and \( f, h, g \in C^1 \)

The necessary conditions for \( x^* \) to be a constrained local minimum are

\[
\begin{align*}
\frac{\partial L}{\partial x} \bigg|_{x^*}^T &= 0^T \\
\eta_i g_i (x^*) &= 0, \ \forall i \\
\eta_i &\geq 0 \quad \forall i
\end{align*}
\]

If the functions \( f \) and \( g \) are convex and the functions \( h \) are linear these conditions are necessary and sufficient!!!
Constrained optimization

Second order **sufficient conditions** for constrained optimality:

Let \( x^* \in D \) and \( f, h, g \in C^2 \) and assume the conditions

\[
\frac{\partial L}{\partial x} \bigg|^{x^*}_T = 0^T \quad \eta_i g_i(x^*) = 0, \eta_i \geq 0 \quad \forall i
\]

\( x^* \) is a strict constrained local minimum if

\[
d^T \frac{\partial^2 L}{\partial x^2} \bigg|^{x^*}_d > 0 \quad \forall d \text{ such that } \frac{dh_i(x)}{dx} \bigg|^{x^*}_d \cdot d = 0, \quad i = 1, \ldots, p
\]
Function spaces

Functional $J : V \rightarrow R$

Vector space $V$, $A \subseteq V$

\[ z^* \in A \] is a local minimum of $J$ over $A$ if there exists an $\varepsilon > 0$ such that for all $z \in A$ satisfying $\|z - z^*\| < \varepsilon$

\[ \Rightarrow J(z^*) \leq J(z) \]
Consider function in $V$ of the form $z + \alpha \eta$, $\eta \in V$, $\alpha \in \mathbb{R}$

The first variation of $J$ at $z$ is the linear function $\delta J \big|_z : V \to \mathbb{R}$ such that $\forall \alpha$ and $\forall \eta$

\[
J(z + \alpha \eta) = J(z) + \delta J \big|_z (\eta) \alpha + o(\alpha)
\]

**First order necessary condition for optimality:**
For all admissible perturbation we must have:

\[
\delta J \big|_{z^*}(\eta) = 0
\]
A quadratic form $\delta^2 J \bigg|_z : V \rightarrow R$ is the second variation of $J$ at $z$ if $\forall \alpha$ and $\forall \eta$ we have:

$$J(z + \alpha \eta) = J(z) + \delta J\bigg|_z (\eta)\alpha + \delta^2 J\bigg|_z (\eta)\alpha^2 + o(\alpha^2)$$

second order necessary condition for optimality: If $z^* \in A$ is a local minimum of $J$ over $A \subset V$ for all admissible perturbation we must have:

$$\delta^2 J\bigg|_{z^*} (\eta) \geq 0$$
The Weierstrass Theorem is still valid

If $J$ is a convex functional and $A \subset V$ is a convex set, a local minimum is automatically a global one and the first order condition are **necessary and sufficient condition** for a minimum.