ESAME 26/1/2015 (1)

ESERCIZIO 1

i) $\nabla f = \begin{pmatrix} 4x_1(x_1^2-2)+2x_1x_2 \\ 2x_2+2+x_1^2 \\ 8x_3^3-1 \end{pmatrix}$ $\nabla^2 f = \begin{pmatrix} 12x_1^2-8+2x_2 & 2x_1 & 0 \\ 2x_1 & 2 & 0 \\ 0 & 0 & 24x_3^2 \end{pmatrix}$

ii) Regione ammissibile: poliedro $\Rightarrow$ S convesso

pb. di min. studio $\nabla^2 f(x)$

$\nabla^2 f(x) \succeq 0 \ \forall x$: NO ad es. $\hat{x}_1 = \hat{x}_2 = 0$ ($\hat{x}_3 = 1$ $\exists$ ammissib.)

$$\nabla^2 f(\hat{x}) = \begin{pmatrix} -8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{INDEFINITO}$$

Non è convesso né concavo.

iii) $\nabla f = 0 \iff \begin{cases} 4x_1^3 - 8x_1 + 2x_1x_2 = 0 \\ 2x_2 + 2 + x_1^2 = 0 \\ x_3^3 = 1/8 \end{cases} \iff \begin{cases} x_1(4x_1^2 - 8 + 2x_2) = 0 \\ x_1^2 = -2-2x_2 \\ x_3 = \frac{1}{2} \end{cases}$

Dalla 1ª $\begin{cases} x_1 = 0 \quad x_2 = -1 \longrightarrow \begin{pmatrix} 0 \\ -1 \\ 1/2 \end{pmatrix} \\ 2x_1^2 - 4 + x_2 = 0 \end{cases}$

sostituisco 2ª $\quad 2(-2-2x_2) - 4 + x_2 = 0$

$-4 - 4x_2 - 4 + x_2 = 0 \quad -3x_2 - 8 = 0$

$$x_2 = -\frac{8}{3}$$

$$x_1^2 = -2 + \frac{16}{3} = \frac{10}{3} \qquad x_1 = \pm\sqrt{\frac{10}{3}}$$

$\hat{x} = \begin{pmatrix} 0 \\ -1 \\ 1/2 \end{pmatrix}$ $\tilde{x} = \begin{pmatrix} \sqrt{10/3} \\ -8/3 \\ 1/2 \end{pmatrix}$ $\bar{x} = \begin{pmatrix} -\sqrt{10/3} \\ -8/3 \\ 1/2 \end{pmatrix}$ studio la natura

$$\nabla^2 f(\hat{x}) = \begin{pmatrix} -10 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$ indefinita si tratta di un punto di sella

$$\nabla^2 f(\check{x}) = \begin{pmatrix} 80/3 & 2\sqrt{10/3} & 0 \\ 2\sqrt{10/3} & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \succ 0$$

minori principali

$\frac{80}{3} > 0$

$\det \begin{vmatrix} 80/3 & 2\sqrt{10/3} \\ 2\sqrt{10/3} & 2 \end{vmatrix} = 40 > 0$

$\det \nabla^2 f(\check{x}) = 3 \cdot 40 > 0$

minimo locale stretto

$$\nabla^2 f(\bar{x}) = \begin{pmatrix} 80/3 & -2\sqrt{10/3} & 0 \\ -2\sqrt{\frac{10}{3}} & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \succ 0$$ come sopra!

minimo locale stretto

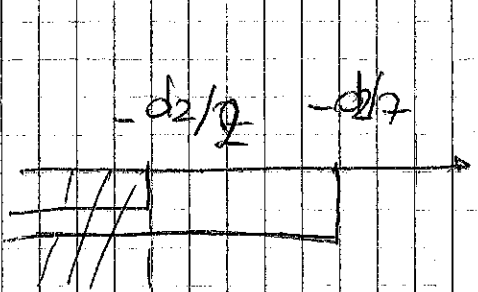
$\exists$ due punti di minimo non vincolato $\check{x}, \bar{x}$

(ii) $\hat{x} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ $\quad \nabla f(\hat{x}) = \begin{pmatrix} 0 \\ 2 \\ 7 \end{pmatrix}$ $\quad I(\hat{x}) = \{1, 3, 4\}$

$$\begin{cases} a_1^T d = 0 \\ a_3^T d \geq 0 \\ a_4^T d \geq 0 \\ \nabla f(\hat{x})^T d < 0 \end{cases} \qquad \begin{cases} d_1 + d_2 + 2d_3 = 0 \\ d_1 \geq 0 \\ d_2 \geq 0 \\ d_2 + 7d_3 < 0 \end{cases} \qquad \begin{cases} d_1 = -d_2 - 2d_3 \\ -d_2 - 2d_3 \geq 0 \\ d_2 \geq 0 \\ d_2 + 7d_3 < 0 \end{cases}$$

$d_2 = 0 \quad \nexists\ d_3$

$d_2 > 0$



$d_3 \leq -d_2/2$

$d_2 \geq 0$

$d_2 = 1 \quad \begin{pmatrix} 0 \\ 1 \\ -1/2 \end{pmatrix} = \hat{d}$

iv) (3)

sist. $\nabla f(x) + \mu_1 \begin{pmatrix}1\\1\\2\end{pmatrix} + \lambda_2 \begin{pmatrix}1\\-1\\-2\end{pmatrix} + \lambda_3 \begin{pmatrix}-1\\0\\0\end{pmatrix} + \lambda_4 \begin{pmatrix}0\\-1\\0\end{pmatrix} + \lambda_5 \begin{pmatrix}0\\0\\-1\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$

non negativ. $\lambda_i \geq 0 \quad i = 2, \dots, 5$

complem. $\lambda_2 (x_1 - x_2 - 2x_3 + 1) = 0$

$\lambda_3 x_1 = 0 \quad \lambda_4 x_2 = 0 \quad \lambda_5 x_3 = 0$

ammiss. $x_1 + x_2 + 2x_3 = 2$

$x_1 - x_2 - 2x_3 \leq -1$

$x_1, x_2, x_3 \geq 0$

$x_1^* = x_3^* = 0 \xRightarrow{\text{AMMISS.}} x_2^* = 2 \qquad \begin{pmatrix}0\\2\\0\end{pmatrix} = x^*$

$I(x^*) = \{1, 3, 5\} \xRightarrow{\text{COMPL.}} \lambda_2^* = 0 \quad \lambda_4^* = 0$

$\nabla f(x^*) = \begin{pmatrix}0\\6\\-1\end{pmatrix}$

$$\begin{cases} 0 + \mu_1 + \cancel{\lambda_2} - \lambda_3 = 0 & \lambda_3 = \mu_1 \\ 6 + \mu_1 - \lambda_2 - \cancel{\lambda_4} = 0 & \lambda_2 = \mu_1 + 6 \\ -1 + 2\mu_1 - 2\lambda_2 - \lambda_5 = 0 & \lambda_5 = -\cancel{2\mu_1} - 12 + \cancel{2\mu_1} - 1 \end{cases}$$

$= -13 < 0$ NO!

non esiste KKT con $x_1 = x_3 = 0$

ESERCIZIO 2

i) $\begin{pmatrix}0\\1\\1/2\end{pmatrix}$ $I(x) = \{1, 2\}$ AMMISS. ma $|I(\bar{x})| < 3$

NO vertice

$\begin{pmatrix}1\\1\\0\end{pmatrix}$ NON AMMISS. (viola secondo vincolo)

$\begin{pmatrix}0\\2\\0\end{pmatrix}$ $I(x)=\{1,3,5\}$ AMMISS $|I(x)|=3$

$rg\begin{pmatrix}1&1&2\\1&0&0\\0&0&1\end{pmatrix}=3$ VERTICE v^1

$\begin{pmatrix}1/2\\1/2\\1/2\end{pmatrix}$ $I(x)=\{1,2\}$ AMMISS ma $|I(x)|=2<n$

NO vertice

$\begin{pmatrix}1/2\\0\\3/4\end{pmatrix}$ $I(x)=\{1,2,4\}$ AMMISS $|I(x)|=3$

$rg\begin{pmatrix}1&1&2\\1&-1&-2\\0&1&0\end{pmatrix}=3$ VERTICE v^2

ii) Scelgo il punto $\begin{pmatrix}1/2\\1/2\\1/2\end{pmatrix}=\bar{x}$

$\begin{cases}a_1^T d=0\\a_2^T d=0\end{cases}$ $\begin{cases}d_1+d_2+2d_3=0\\d_1-d_2-2d_3=0\end{cases}$ $\Rightarrow d_1=0$

$d_2+2d_3=0$ $d_2=-2d_3$ $\bar{d}=\begin{pmatrix}0\\-2d_3\\d_3\end{pmatrix}$ $d_3\in\mathbb{R}$

$\bar{d}=\pm\begin{pmatrix}0\\-2\\1\end{pmatrix}t$

Considero $j\notin I(\bar{x})$ e verifico se $\exists j$ t.c. $a_j^T\bar{d}<0$

$a_3^T\bar{d}=0$

per $d_3>0$ attivo vincolo 4°

$a_4^T\bar{d}=-2d_3$

per $d_3<0$ attivo vincolo 5°

$a_5^T d=d_3$

Scelgo $d_3>0$ $\bar{d}=\begin{pmatrix}0\\-2\\1\end{pmatrix}$ $t^{max}\Longrightarrow$

$$y = \bar{x} + t\bar{d} = \begin{pmatrix} 1/2 \\ 1/2 - 2t \\ 1/2 + t \end{pmatrix} \qquad t^{max} : \frac{1}{2} - 2t \geq 0 \quad t \leq \frac{1}{4}$$

$$t^{max} = 1/4$$

$$y = \begin{pmatrix} 1/2 \\ 0 \\ 3/4 \end{pmatrix}$$

iii)

$$\max\ x_1 + 2x_2 - 6x_3$$
$$x_1 + x_2 + 2x_3 = 2$$
$$-x_1 + x_2 + 2x_3 - x_4 = 1$$
$$x_i \geq 0$$

iv)

$$v_1 = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} \qquad SBA^1 = \begin{pmatrix} 0 \\ 2 \\ 0 \\ 1 \end{pmatrix} \rightarrow \begin{array}{l} x_B = \begin{pmatrix} x_2 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \\ x_N = \begin{pmatrix} x_1 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{array}$$

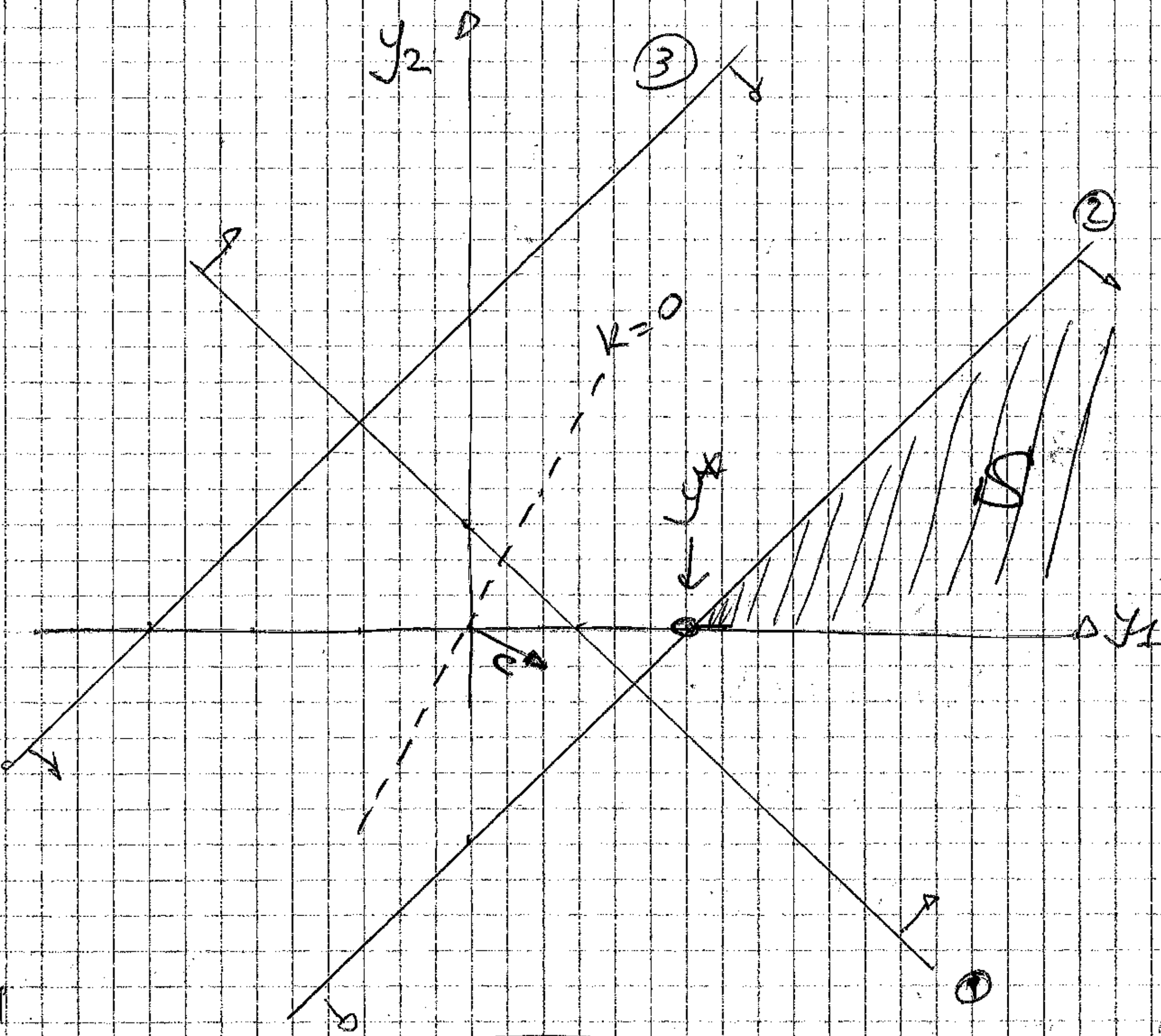
$$B = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} \quad N = \begin{pmatrix} 1 & 2 \\ -1 & 2 \end{pmatrix}$$

$$v^2 = \begin{pmatrix} 1/2 \\ 0 \\ 3/4 \end{pmatrix} \qquad SBA^2 = \begin{pmatrix} 1/2 \\ 0 \\ 3/4 \\ 0 \end{pmatrix}$$

ESERCIZIO 3

i)

$$\min\ 2y_1 - y_2$$
$$y_1 + y_2 \geq 1$$
$$y_1 - y_2 \geq 2$$
$$2y_1 - 2y_2 \geq -6$$
$$y_2 \geq 0$$



$b^T y^* = 4$

$y^* = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$ $\boxed{I(y^*) = \{2,4\} \Rightarrow x_1^* = x_3^* = 0}$

iii) COMPL. PRIM. e DUALE + AMMISS.

$y_2 \, (x_1 - x_2 - 2x_3 + 1) = 0$

$x_1 \, (1 - y_1 - y_2) = 0$

$x_2 \, (2 - y_1 + y_2) = 0$

$x_3 \, (-6 - 2y_1 + 2y_2) = 0$

AMMISS. PRIMALE $x_2^* = 2 \Rightarrow \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} = x^*$

$c^T x^* = 4$

OK

iv) Dall'analisi precedente $c^T x^*_\varepsilon = b^T y^* + \varepsilon y_1^* =$

$= 4 + \varepsilon 2$

$c^T x^*_\varepsilon - c^T x^* = 2\varepsilon$

v) Simmetricamente sul duale $c_2 \to c_2 + \varepsilon$ corrisponde a modificare vincolo.

$b^T y^*_\varepsilon = c^T x^* + \varepsilon x_2^*$ $\quad b^T y^*_\varepsilon - b^T y^* = 2\varepsilon$

vi) La soluzione ottima rimane tale poiché già soddisfa il vincolo $x^* \in \mathbb{Z}^n$.

ESERCIZIO 4

Variabili di decisione x_i = q.tà Lettori mod. i prodotti $i = 1,2,3$

F. obiettivo $\max \text{profitto} = \sum_{i=1}^{3} \text{prezzo}_i \cdot x_i$

VINCOLI: $x_i \geq 0 \quad \forall i = 1,2,3$

Indicati con t_{ij} tempo lavorazione PRODOTTO i su macchina j e M_j = ore disponibili macch. j

$$\sum_{i=1}^{3} t_{ij} x_i \leq M_j \quad \forall j = 1,2,3,4$$

$$x_1 \leq 0.45 \sum_{i=1}^{3} x_i$$

$$x_2 \leq 0,35 \sum_{i=1}^{3} x_i$$

$$x_3 \leq 0.40 \sum_{i=1}^{3} x_i$$