# Chapter 10

# Integer Linear Programming

The Integer Linear Programming (ILP) is the problem of minimization (maximization) of a linear function subject to equality and inequality constraints and the restriction that one or more variables can take only integer values. It is therefore problems of the type Pure ILP when

$$\begin{array}{ll} \min & c^T x \\ & Ax \geq b \\ & x \in Z^n; \end{array}$$

or Mixed ILP

$$\begin{array}{ll} \min & c^T x \\ & Ax \geq b \\ & x_1 \in Z^{n_1}, x_2 \in R^{n_2}; \end{array}$$

con $x = (x_1\ x_2)$ e $n_1 + n_2 = n$.

If the feasible region of the ILP is made up of a finite set of points, in principle, it could be always possible to solve the problem by calculating the value of the objective function at each feasible point and choosing the one that minimizes the objective function. We will define this method  bf full enumeration. When the number of integer feasible points is small, full enumeration is not only possible, but it may be the best way to solve a ILP problem. Vice versa, if the cardinality is large, the full enumeration total becomes unfeasible.

We observe that if we have an ILP problem with ten variables and each of these variables can take ten different values, the number of possible integer feasible points is $10^{10}$, i.e. 10 billion. This simple example shows that the full enumeration is rarely useful in practical cases, where often the number of integer variables is of the order of hundreds if not thousands.

## 10.1 Classic use of Integer variables

### 10.1.1 The binary knapsack problem

Suppose you have $n$ projects. The $j$-th project $j = 1, \ldots, n$, has a cost of $a_j$ and a value of $c_j$. Each selected project must be either selected or not. Finally there is a budget $b$ on the maximun cost sustainable. The problem of choosing a subset of projects in order to maximize the sum of values without exceeding the limit imposed by the ”budget” is the so-called it knapsack binary problem.

- it variables. To each project $i = 1,\dots,n$ we introduce the following binary variables

$$x_i = \begin{cases} 1 & \text{if project } i \text{ selected} \\ 0 & \text{otherwise.} \end{cases}$$

The budget constraint is $\sum_{j=1}^{n} a_j x_j \le b$ and the problem is

$$\begin{aligned} \max \quad & \sum_{j=1}^{n} c_j x_j \\ & \sum_{j=1}^{n} a_j x_j \le b \\ & x_j \in \{0,1\} \quad j = 1,\dots,n. \end{aligned}$$

An example of knapsack problem is the CAPITAL BUDGETING discussed during the first lectures.

### 10.1.2 Assignment

An assignment problem consists in finding the optimal way to assign jobs to individuals or, more generally, to assign resources (people, machines, etc.) to activities.
You can see slide of the first lecture for a description of the problem.

## 10.2 Use of Boolean variables to model logical conditions

See the Chapter 0.

## 10.3 Relationship among LP and ILP

IN this section we consider

$$\begin{aligned} z_I^* \quad = \min \quad & c^T x \\ & Ax \ge b \\ & x \;\; \text{intera} \end{aligned} \tag{PLI}$$

We denote by $S_I = \{x \in Z^n : Ax \le b\}$ the feasible region of the ILP and with $x_I^* \in S_I$ an optimal solution namely $c^T x_I^* = z_I^*$.
The LP problem obtained by eliminating the integrality constraints in the ILP problem is called *linear/continuous relaxation*

$$\begin{aligned} z^* \quad = \min \quad & c^T x \\ & Ax \ge b \end{aligned} \tag{PL}$$

We denote by $S$ the polyhedron of the conitnuous relaxation $S = \{x \in R^n : Ax \leq b\}$ and with $x^* \in S$ the optimal solution of the LP relaxation such that $c^T x^* = z^*$ for all $x \in S$.
Of cure $S_I \subset S$ so that the optinal value of the LP relaxation is always not worst of the ILP one; in the case of minimization problems

$$z^* \leq z_I^*,$$

so that optimal value n the solution of the linear relaxation constitutes a lower bound to the optimal value of the ILP problem.

Given two sets $S' \subseteq S$ it holds

$$\min_{x \in S'} f(x) \geq \min_{x \in S} f(x).$$

.

If an optimal solution of the LP relaxation $x^* \in S$ is integer then $x^* \in S_I$ and hence it is also the optimal solution of ILP, i.e. $x^* = x_I^*$.

On the hand, if any component of $x^* \in S$ is not integer then $x^*$ is unfeasbile for the ILP.

You may think of finding a solution by rounding $x^*$ to the nearest integer solution. *Rounding* can work in practice in a lot of cases in which integer variables represent number of items and they take large optimal values. of course it works if feasibility is preserved by rounding.

However if the optimal value of the integer variables is small (few units) that this approach can cause large errors.

The situation becomes even more dramatic when Integer variables are binary variables. In these problems the 0-1 variables indicate which of two possible alternatives must be implemented. In this case rounding is meaningless.

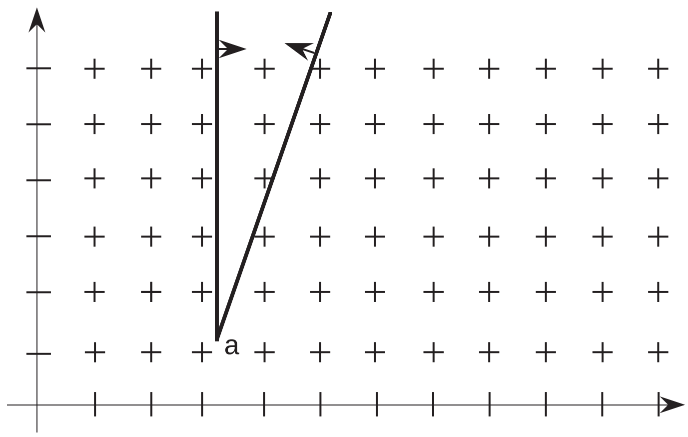


Figure 10.1: Rounding: a first example.

As a matter of example, consider figure 10.1. Assume that the solution of the LP relaxation is $\bar{x} = a$. The point $a$ is not integer and rounding (either floor or ceiling) does not give a integer feasible solution.

even in the case in which it is posisble to find an integer value “near” $\bar{x}$, this ca be far from the the optimal ILP solution.

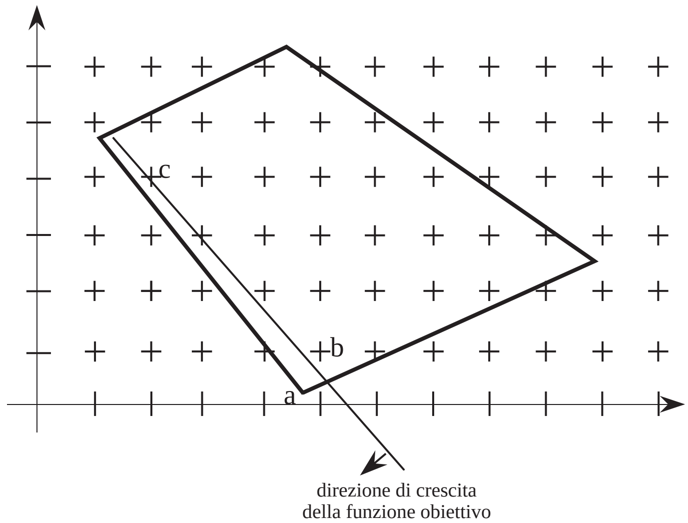


Figure 10.2: Rounding: a second example.

Such an example is in figure 10.2, where the LP relation has a solution in *a* and *b* is the integer feasible point obtained by rounding, whereas *c* is the true integer solution.

**Esempio 8** *Consider the problem*

$$\begin{array}{ll} \min & -x_1 - x_2 \\ & -2x_1 + 5x_2 \leq 5 \\ & 2x_1 - 2x_2 \leq 1 \\ & x \geq 0,\ \textit{intero} \end{array}$$

*whose integer solutions are*

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

*The optimal one is* $x_I^* = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ *with value* $z_I^* = -2$.
*Solving the linear relaxation*

$$\begin{array}{ll} \min & -x_1 - x_2 \\ & -2x_1 + 5x_2 \leq 5 \\ & 2x_1 - 2x_2 \leq 1 \\ & x \geq 0 \end{array} \tag{10.1}$$

*we get the continuous solution* $x_1^* = \frac{5}{2} = 2.5, x_2^* = 2$ *with value* $z^* = -\frac{9}{2}$. *By rounding to the upper or lower integer we get the solutions* $\begin{pmatrix} 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ *which are both not feasible. In figurw 10.3 the re is the plot of the feasible regions* $S$ *e* $S_I$.

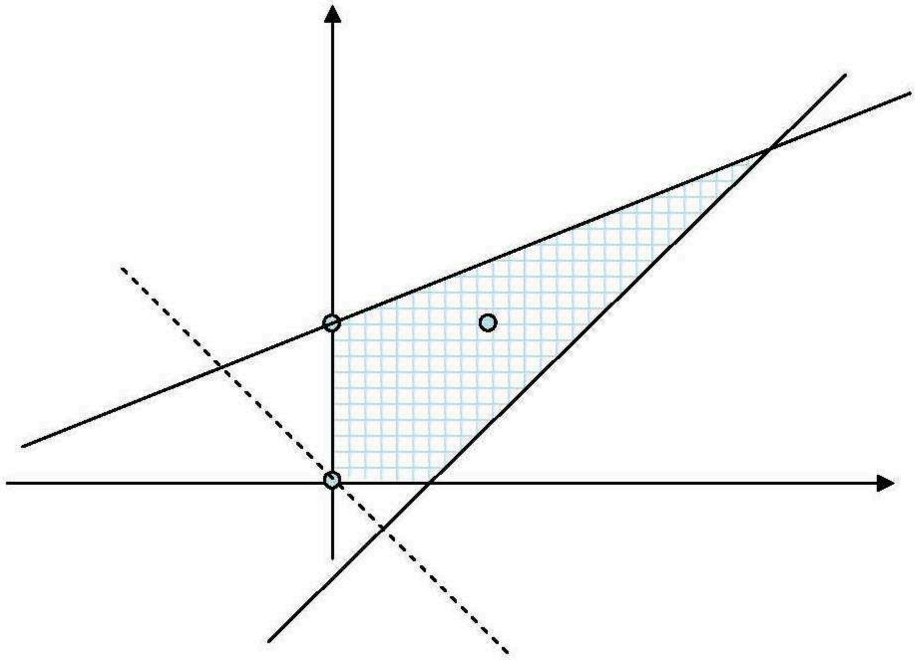

Figure 10.3: Graphical solution of Example 10.1.

Hence rounding to an integer value is not always a good strategy. Further rounding can be computationally heavy. Indeed given a fractional solution $\bar{x} \in \Re^n$, there are $2^n$ possible integer vector obtainable by upper or lower rounding the not integer components of $\bar{x}$. Many if these $2^n$ vector can be not feasible for the ILP problem and founding a feasible integer solution"not to far" from the fractional $\bar{x}$ can be in general as difficult as the original ILP problem.

We note however that the same ILP problem can be formulated in different equivalent ways.

**Esempio 8 (continue)** *Consider again example 8 with the additional constraint* $x_1 \leq 1$

$$\begin{array}{ll} \min & -x_1 - x_2 \\ & -2x_1 + 5x_2 \leq 5 \\ & 2x_1 - 2x_2 \leq 1 \\ & x_1 \leq 1 \\ & x \geq 0, \ integer \end{array} \tag{10.2}$$

*The feasible region $S_I$ remains the same, whereas the feasible region of the relaxed problem and its optimal solution change (see the picture 10.4).*

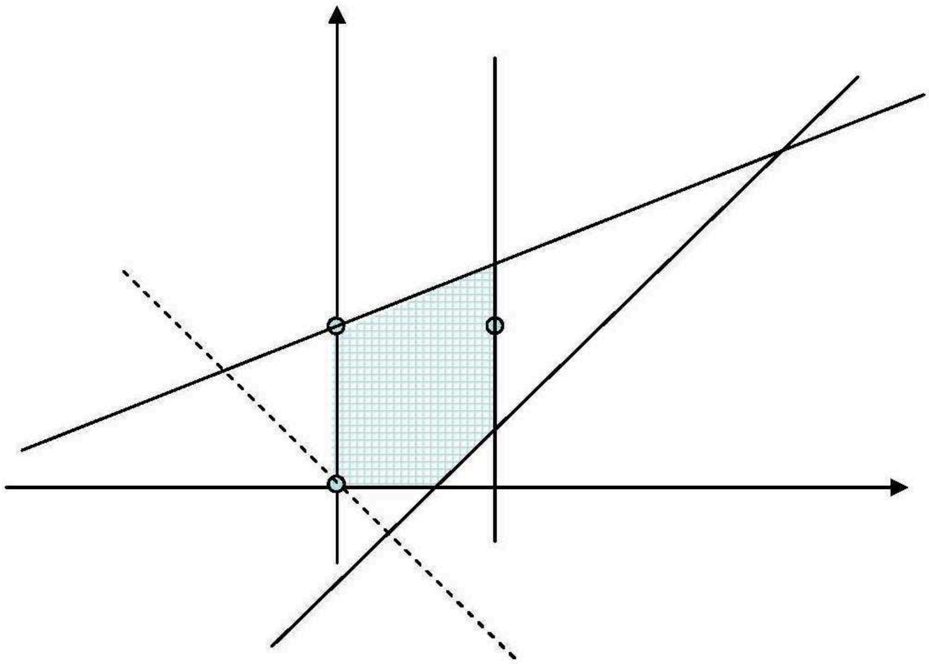

Figure 10.4: Graphical solution of Example 10.2.

*Consider the problem*

$$\begin{array}{ll} \min & -x_1 - x_2 \\ & -2x_1 + 5x_2 \leq 5 \\ & 2x_1 - 2x_2 \leq 2 \\ & x_2 \leq 1 \\ & x \geq 0, \ \textit{intero} \end{array} \tag{10.3}$$

*with the feasible region reported in the picture 10.5. The solution of the linear relaxation coincides $x_I^*$. This last formulation has a special property that the solution of the relaxation coincide with the solution of the integer problem. It is an "ideal" formulation.*

Obviously the "ideal" formulation is not known in general, because if it were known, the ILP could be solve by linear programming. However thee are problems that possess this property

**Definition 10.4** *A polyhedron is called* integer *when al the vertex are integer.*

## 10.5 Integer property and unimodularity

For sake of simplicty we start the study of integer propertie with reference to the standard polyhedron $P = \{x \in \mathbb{R}^n : Ax = b, \ x \geq 0\}$. We proved in section 9.1.1 a one toone correspondence among vertex of the polyhedron and BFS, $x_B = B^{-1}b \in R^m$, $x_N = 0$. We would like toget a condition under which $B^{-1}b$ is integer for any choice of the basis $B$.

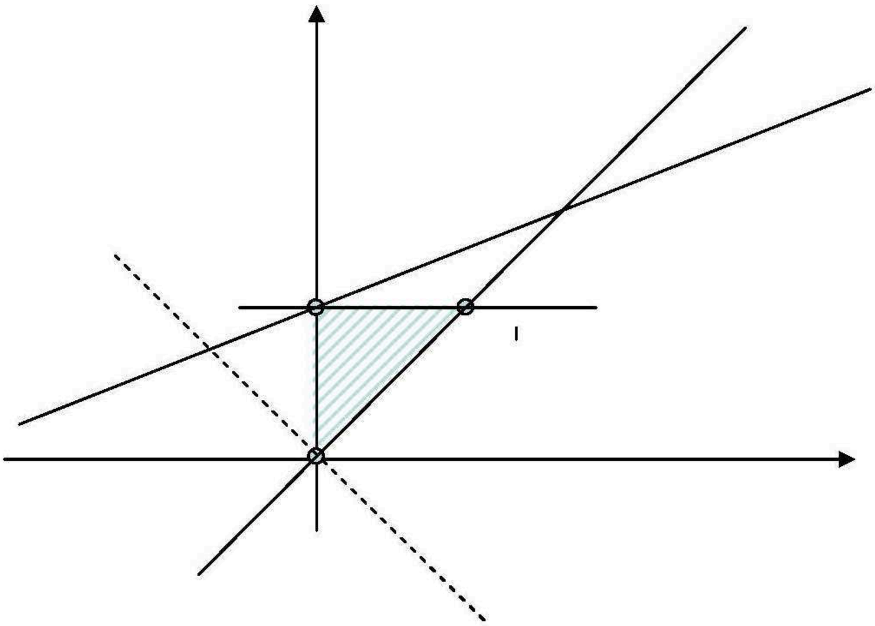

Figure 10.5: Graphical solution of Example 10.3.

**Theorem 10.6** *if a square matrix B with integer components has determinat equal to* $\pm 1$*, then its inverse* $B^{-1}$ *is integer.*

The proof is trivial recalling the construction of the inverse matrix equal tothe adjoint matrix (integer for integer $B$) divided by the determinant.

**Definition 10.7** *A matrix A* $(m \times n)$ *with integer components and rank equal to m is called* unimodular *if any* $m \times m$ *submatrix of A has determinant equal to* $-1, 1$.

Hence if $A$ $(m \times n)$ with integer components unimodular, then for any integer $b$ the polyhedron

$$P = \{x \in \mathbb{R}^n : Ax = b,\ x \geq 0\}$$

has integer vertexes.

We have also a stronger result.

Let $A$ a $m \times n$ matrix with integer components and rank equal to $m$. The polyhedron

$$P = \{x \in \mathbb{R}^n : Ax = b,\ x \geq 0\}$$

has integer vertexes for any integer $b$ if and only if $A$ is unimodular.

A matrix $A$ is called *totally unimodular* if any square submatrix of $A$ has determinant in $\{-1, 0, 1\}$.

**Theorem 10.8** *Let A a $m \times n$ matrix with integer components. The polyhedron*

$$P = \{x \in \mathbb{R}^n : Ax \geq b,\ x \geq 0\}$$

*has integer vertexes for any integer b if and only if A is totally unimodular.*

We can give conditions to check wether a matrix is totally unimodular.

**Theorem 10.9** *Let A a $m \times n$ matrix with elements $a_{ij} \in \{0, 1, -1\}$. If*

1. *each column has at most two elements different from zero;*
2. *it is possible to partition the row indexes in the subsets $Q_1$ e $Q_2$ such that*
   (i) *if the column $j$ has two elements $a_{ij} \neq 0$ and $a_{kj} \neq 0$ with same sign then $i \in Q_1$ and $k \in Q_2$;*
   (ii) *if the column $j$ has two elements $a_{ij} \neq 0$ and $a_{kj} \neq 0$ with different sign then $i, k \in Q_1$ or $i, k \in Q_2$.*

*Then the matrix A is totally unimodular.*

**Esempio 9** *Check that the matrix*

$$\begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

*does nor satisfy the sufficient condition but it is totally unimodular.*

We can give a stronger result for matrixes with at most two elements different from zero.

**Theorem 10.10** *Let A a $m \times n$ matrix with elements $a_{ij} \in \{0, 1, -1\}$ and such that each column has at most two elements different from zero. The matrix A is totally unimodular if and only if it is possible to partition the row indexes in the subsets $Q_1$ e $Q_2$ such that*

(i) *if the column $j$ has two elements $a_{ij} \neq 0$ and $a_{kj} \neq 0$ with same sign then $i \in Q_1$ and $k \in Q_2$;*

(ii) *if the column $j$ has two elements $a_{ij} \neq 0$ and $a_{kj} \neq 0$ with different sign then $i, k \in Q_1$ or $i, k \in Q_2$.*

We also have

**Theorem 10.11** *If A is totally unimodular then so they are also the matrices*

$$\begin{pmatrix} A \\ I \end{pmatrix} \quad \begin{pmatrix} A \\ -I \end{pmatrix} \quad \begin{pmatrix} A & I \end{pmatrix} \quad \begin{pmatrix} A & -I \end{pmatrix}$$

**Esempio 10** *Check if the matrix is totally unimodular*

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \end{pmatrix}$$

**Solution.**
All the elements are in $0,1$ and each column has at most two elements different from zero. We can try to construct the partition of the row indexes. If row $1 \in Q_1$ then row $2 \in Q_2$, but row 3 cannot belong neither to $Q_1$ (due to $a_{12}$) nor in $Q_2$ (due to $a_{13}$). Hence the matrix is NOT totally unimodular. □

**Esempio 11** *Check if the matrix is totally unimodular*

$$\begin{pmatrix} 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

**Solution.**
All the elements are in $0,1$ and each column has at most two elements different from zero. We can construct the partition of the row indexes $Q_1 = \{1,2,3\}$, $Q_2 = \{4,5,6\}$, so that the matrix is totally unimodular. □

**Esempio 12** *Check that the matrix of the assignment problem is totally unimodular.*

**Esempio 13** *Check that the matrix of the transportation problem is totally unimodular.*

**Esempio 14** *Check that the matrix of the revenue management problem is totally unimodular.*

## 10.12 Solution of ILP problems

Consider the ILP

$$\begin{aligned} \min \quad & c^T x \\ & Ax \le b \\ & 0 \le x \le U \\ & x \quad \text{integer} \end{aligned} \tag{ILP}$$

where $U$ is an upper bound value on the variables. We use explicitly this constraint to ensure that *the feasible region if ILP problem s bounded.*

We define the *linear relaxation of the ILP problem* as

$$\begin{aligned} \min \quad & c^T x \\ & Ax \le b \\ & 0 \le x \le U. \end{aligned} \qquad (PR)$$

It is an LP problem obtained by the ILP by removing the integrality constraint. The feasible region of ILP is contained in the feasible region of the LP relaxed problem and hence the optimal value of (PR) is not worst of that of ILP. In particular, if the optimal solution od (PR) is integer than it is an optimal solution of the ILP.

### 10.12.1 Branch and Bound method

The "Branch and Bound" (BB) is a method for the solution of ILP problems that tries to partially explored the feasible region. Let $S^{(0)}$ the set of feasible solutions of ILP problem (that will be denoted by $P^{(0)}$). A *partition* of $S^{(0)}$ is a family of subsets $(S^{(1)},\dots,S^{(r)})$, $r \ge 2$ such that:

$$S^{(i)} \cap S^{(j)} = \emptyset \qquad \text{for each pair} \qquad 1 \le i < j \le r$$

and

$$\bigcup_{i=1}^{r} S^{(i)} = S^{(0)}$$

Obviously an optimal solution of the ILP is obtained as the minimum among the values $z^{(1)^*} = c^T x^{(1)^*},\dots,z^{(r)^*} = c^T x^{(r)^*}$ where $x^{(i)^*}$ is an optimal solution of problem ILP$_i$ defined as the minimization of the objective function over $S^{(i)}$ (in the following we denote by $\mathscr{P}^{(i)}$ the subproblem PLI$_i$). If a given PLI$_i$ turns out to be difficult to be solved, its feasible set $S^{(i)}$ can be further partitioned obtaining new subproblems. The procedure can be iterated as long as we end with easily solvable problems.

Let's consider a trivial example. Assume that we want to find the youngest student of Sapienza. Of course we can solve the problem by comparing the age of all the students in Sapienza (total enumeration) and this can be a long process. Otherwise we may ask to the dean of each faculty for the age of their youngest student and then choose the youngest among those selected. We have partitioned (each student belongs only to a faculty) the problem ito smaller subproblems. When the faculty is small, the solution of this subproblem may be easy, but in other cases the dean may subdivided the problem in turn and ask to each responsible of a course the age of the youngest student. And so on.

This procedure may be efficient if the number of subproblems generated is small and bounded and the solution of each subproblem is easier and efficient. However in some cases the subproblem $P^{(i)}$ is not solved exactly nd only a **"lower bound"** of $z^{(i)^*}$ namely a value $L^{(i)} \le z^{(i)^*}$ is found. This value is compared with the current best known value $\tilde{z}$ of the objective function is a feasible integer point (**incumbent or current optimal**). If we have

$$\tilde{z} \le L^{(i)}$$

since $L^{(i)} \le z^{(i)*}$ we can assert that in the set $S^{(i)}$ there exists no integer point where the objective function gets a better value than $\tilde{z}$. Hence we can stop solving problem $P^{(i)}$ that can be eliminated from the list of subproblems to be solved.

Using again the trivial example, we can assume that one of the dean stated that one student is 17 years and 3 month (incumbent). Assume that another dean, even without knowing the age of the youngest student, knows that in her/his faculty there is no younger student than 18 years. This means that the youngest student of Sapienza cannot belong to this faculty (because we know that there is a younger student) so that it is not worthwhile for the dean to put effort in finding the exact age of the youngest student.

Of course efficiency in the solution of the original problem depends both on good values of the "lower bound" and on the decomposed problem. Hence there are two main strategies that play a fundamental role in defining a B&B algorithm. These are the

**(a) Solution strategy**, namely how to find a lower bound.

**(b) Separation strategy**, namely the way to partition the feasible set.

Starting from problem ILP $P^{(0)}$, we consider subproblems $P^{(i)}$ of the type

$$\begin{array}{ll} \min & c^T x \\ & Ax \le b \\ & l^{(i)} \le x \le u^{(i)} \\ & x \ \text{ integer}, \end{array} \qquad (P_i)$$

where the vectors $l^{(i)}$ and $u^{(i)}$ are defined within the algorithm. We denote by $x_I^{(i)*}$ and $z^{(i)*}$ an optimal solution and the corresponding optimal value of $P_i$.

In order to get the lower bound $L^{(i)}$ of $z^{(i)*}$ we consider *the linear relaxation of problem* $P^{(i)}$. We denote by $x_R^{(i)}$ the optimal solution of the linear relaxation of $P^{(i)}$ and by $L^{(i)} = c^T x_R^{(i)}$ the corresponding optimal value. By definition $L^{(i)} \le z^{(i)*}$, and the linearrelaxation can be solved e.g. by the simplex method.

The separation strategy is based on the solution of the relaxation of the problem $P^{(i)}$. Let $x_R^{(i)}$ the optimal solution of the relaxed problem and $L^{(i)}$ the corresponding optimal value. If $x_R^{(i)}$ has all integer components then $x_R^{(i)} = x_I^{(i)*}$ and we have also the solution of $P^{(i)}$.

If $x_R^{(i)}$ has some fractional components and if $L^{(i)}$ is less or equal than the *incumbent*, then the integer optimal solution of $P^{(i)}$ cannot be better than the incumbent, so that we need not to further separate $P^{(i)}$.

Let $x_{Rj}^{(i)}$ a fractional component for the vector $x_R^{(i)}$, define $\alpha_j = \lfloor x_{Rj}^{(i)} \rfloor$ the largest integer value less or equal to $x_{Rj}^{(i)}$ e $\beta_j = \lceil x_{Rj}^{(i)} \rceil$ the smallest integer grater or equal to $x_{Rj}^{(i)}$. Of course

$$\begin{array}{l} \alpha_j < \beta_j \\ \beta_j - \alpha_j = 1. \end{array}$$

We partition problem $S^{(i)}$ in the two subproblems

$$\begin{array}{ll} \min & c^T x \\ & Ax \leq b \\ & (l^{(i)})^1 \leq x \leq u^{(i)} \\ & x \ \text{ integer,} \end{array} \qquad (l^{(i)})^1_j = \begin{cases} l^{(i)}_j & \text{if } i \neq j \\ \beta_j & \text{if } i = j \end{cases}$$

e

$$\begin{array}{ll} \min & c^T x \\ & Ax \leq b \\ & l^{(i)} \leq x \leq (u^{(i)})^2 \\ & x \ \text{ integer,} \end{array} \qquad (u^{(i)})^2_j = \begin{cases} u^{(i)}_j & \text{if } i \neq j \\ \\ \alpha_j & \text{if } i = j \end{cases}$$

The last case to be considered is when the feasible region of the relaxation of $P^{(i)}$ is empty. since the feasible region $S^{(i)}$ is contained into the relaxed one this implies that also $S^{(i)}$ is empty, so that we can close problem $P^{(i)}$.

The B&B scheme is reported below

**1. Initialization**

Solve the relaxation of problem $P^{(0)}$ obtaining the "lower bound" $L^{(0)}$.

Find a first **"upper bound"** $\tilde{z}$ for problem $P^{(0)}$. This UB value can be found by evaluating the objective function at a feasible integer solution $\tilde{x}$, when it is easy to be found. When such an integer solution cannot be easily found, set $\tilde{z} = \infty$.

If the optimal relaxed solution $x_R^{(0)}$ is in $S^{(0)}$, the problem is solved. Otherwise branch $S^{(0)}$ obtaining $S^{(1)}$ and $S^{(2)}$. Obtain the "lower bounds" $L^{(1)}$ and $L^{(2)}$. If $S^{(1)}$ (or $S^{(2)}$) is *empty* then you can set $L^{(1)}$ ($L^{(2)}$) to the value $\infty$ (which implies that the problem is eliminated from the list).

**2. Generic step.**

Assume the following list of problems is given $L = \{P^{(1)}, \dots, P^{(q)}\}$ and for each problem $P^{(i)}$ let $L^{(i)}$ be a "lower bound".

*2.1. Closing of subproblem dominated by the incumbent*

If $L^{(i)} \geq \widetilde{z}$, then the optimal solution $(z^{(i)})^*$ of problem $P^{(i)}$ satisfies:

$$z^{(i)^*} \geq L^{(i)} \geq \widetilde{z}$$

Hence there is no better solution than $\tilde{z}$ in $S^{(i)}$ so that $P^{(i)}$ can be closed and eliminated from the list of open problems. We can assume that all the open problems are such that $L^{(i)} > \tilde{z}$. *If all the problems in the list are closed the algorithm stops and the optimal solution is the incumbent $\tilde{z}$.* Otherwise we proceed as follows

*2.2. Choose of a problem from the list*

We have different strategies, among which:

**(1)** *greatest "lower bound".*

**(2)** *LIFO (Last In First Out).*

*2.3. Analysis of a subproblem*

Let $P^{(i)}$ the chosen problem and $L^{(i)}$ the corresponding "lower bound" obtained at the optimal solution $x_R^{(i)}$ of the relaxation $L^{(i)} = c^T x_R^{(i)}$. Then we can have:

*2.3.1.* $x_R^{(i)} \in S^{(i)}$

If $\bar{x}^{(i)} \in S^{(i)}$ namely it is integer hence also optimal for $P^{(i)}$. Further, since $P^{(i)}$ is an open problem $L^{(i)} < \tilde{z}$ and hence $x_R^{(i)}$ has a better value $c^T x_R^{(i)} = L^{(i)} < \tilde{z}$. We set $\tilde{x} = \bar{x}^{(i)}$ and $\tilde{z} = L^{(i)}$. (**update of the incumbent**) and problem $P^{(i)}$ is closed.

*2.3.2.* $\bar{x}^{(i)} \notin S^{(i)}$

We apply the branching strategy described above. □

ATTENTION: if the problem is a *maximization* problem instead of *minimization*, namely

$$\begin{array}{ll} \max & c^T x \\ & Ax \le b \\ & 0 \le x \le U \quad x \ \text{ integer} \end{array}$$

The optimal value of the relaxation is an "upper bound" $U$ and we have that $z_I^* \le U$. Any integer feasible solution is a "lower bound"=incumbent.

### 10.12.2 Examples

**Example 1.** Consider the Maximization problem

$$\begin{array}{lll} (P^{(0)}) & \max & 3x_1 + x_2 \\ & & 7x_1 + 2x_2 \le 22 \\ & & -2x_1 + 2x_2 \le 1 \\ & & 1 \le x_1 \le 4 \\ & & 0 \le x_2 \le 3 \\ & & x_1, x_2 \in \mathscr{Z} \end{array}$$

The optimal solution of the relaxation is $x_R^{(0)} = (7/3, 17/6)^T$ and gives an "upper bound" $U^{(0)} = z(x_R^{(0)}) = 59/6 = 9.8333\ldots$.

By flooring $x_R^{(0)}$ we get ${}^T\tilde{x} = (2,2)$ which is feasible so that we have the incumbent ("lower bound") $\tilde{z} = z(\tilde{x}) = 8$. The list of open problems is $\mathscr{L} = \{S^{(0)}\}$.

we branch over $P^{(0)}$. We can choose any of the fractional variables. We choose $x_1$ and we get the two subproblems

$$
\begin{array}{lll}
(P^{(1)}) & \max & 3x_1 + x_2 \\
& & 7x_1 + 2x_2 \leq 22 \\
& & -2x_1 + 2x_2 \leq 1 \\
& & 1 \leq x_1 \leq 2 \\
& & 0 \leq x_2 \leq 3 \\
& & x_1, x_2 \in \mathscr{Z}
\end{array}
$$

$$
\begin{array}{lll}
(P^{(2)}) & \max & 3x_1 + x_2 \\
& & 7x_1 + 2x_2 \leq 22 \\
& & -2x_1 + 2x_2 \leq 1 \\
& & 3 \leq x_1 \leq 4 \\
& & 0 \leq x_2 \leq 3 \\
& & x_1, x_2 \in \mathscr{Z}
\end{array}
$$

The linear relaxation of $P^{(1)}$ has optimal solution $x_R^{(1)} = (2, 5/2)^T$ with value $U^{(1)} = z(x_R^{(1)}) = 8.5$. The linear relaxation of $P^{(2)}$ has optimal solution $x_R^{(2)} = (3, 1/2)^T$ with value $U^{(2)} = z(x_R^{(2)}) = 9.5$.

Both problem cannot be closed and the list is $\mathscr{L} = \{P^{(1)}, P^{(2)}\}$.

We choose $P^1$ and branch with respect to the only fractional component $x_2$. We get the subproblems

$$
\begin{array}{lll}
(P^{(3)}) & \max & 3x_1 + x_2 \\
& & 7x_1 + 2x_2 \leq 22 \\
& & -2x_1 + 2x_2 \leq 1 \\
& & 1 \leq x_1 \leq 2 \\
& & 0 \leq x_2 \leq 2 \\
& & x_1, x_2 \in \mathscr{Z}
\end{array}
$$

$$
\begin{array}{lll}
(P^{(4)}) & \max & 3x_1 + x_2 \\
& & 7x_1 + 2x_2 \leq 22 \\
& & -2x_1 + 2x_2 \leq 1 \\
& & 1 \leq x_1 \leq 2 \\
& & 3 \leq x_2 \leq 3 \\
& & x_1, x_2 \in \mathscr{Z}
\end{array}
$$

The linear relaxation of $P^{(3)}$ has optimal solution $x_R^{(3)} = (2,2)^T$ with value $U^{(3)} = z(x_R^{(3)}) = 8$. The linear relaxation of $P^{(4)}$ is unfeasible so that we pose $U^{(4)} = -\infty$.

We can close $P^{(3)}$ because the solution is integer. We do not update the incumbent. We can close $P^{(4)}$ because it is unfeasible.

Now the list is $\mathscr{L} = \{P^{(2)}\}$. We select $P^{(2)}$ and branche with respect to $x_2^{(2)}$. We obtain the subproblems

$$
\begin{array}{lll}
(P^{(5)}) & \max & 3x_1 + x_2 \\
 & & 7x_1 + 2x_2 \leq 22 \\
 & & -2x_1 + 2x_2 \leq 1 \\
 & & 3 \leq x_1 \leq 4 \\
 & & 0 \leq x_2 \leq 0 \\
 & & x_1, x_2 \in \mathscr{Z}
\end{array}
$$

$$
\begin{array}{lll}
(P^{(6)}) & \max & 3x_1 + x_2 \\
 & & 7x_1 + 2x_2 \leq 22 \\
 & & -2x_1 + 2x_2 \leq 1 \\
 & & 3 \leq x_1 \leq 4 \\
 & & 1 \leq x_2 \leq 3 \\
 & & x_1, x_2 \in \mathscr{Z}
\end{array}
$$

The linear relaxation of $P^{(5)}$ has optimal solution $x_R^{(5)} = (3.14, 0)^T$ with value $U^{(5)} = z(x_R^{(5)}) = 9.428$. The linear relaxation of $P^{(6)}$ is unfeasible so that we pose $U^{(6)} = -\infty$.

Problem $P^{(6)}$ can be closed whereas $P^{(5)}$ enters the list $\mathscr{L}$ because $U^{(5)} > \tilde{z}$.

We select $P^{(5)}$ and branch w.r.t. $x_1$ obtaining

$$
\begin{array}{lll}
(P^{(7)}) & \max & 3x_1 + x_2 \\
 & & 7x_1 + 2x_2 \leq 22 \\
 & & -2x_1 + 2x_2 \leq 1 \\
 & & 3 \leq x_1 \leq 3 \\
 & & 0 \leq x_2 \leq 0 \\
 & & x_1, x_2 \in \mathscr{Z}
\end{array}
$$

$$
\begin{array}{lll}
(P^{(8)}) & \max & 3x_1 + x_2 \\
 & & 7x_1 + 2x_2 \leq 22 \\
 & & -2x_1 + 2x_2 \leq 1 \\
 & & 4 \leq x_1 \leq 4 \\
 & & 0 \leq x_2 \leq 0
\end{array}
$$

$$x_1, x_2 \in \mathscr{Z}$$

The linear relaxation of $P^{(7)}$ has optimal solution $x_R^{(7)} = (3,0)^T$ with value $U^{(7)} = z(x_R^{(7)}) = 9$. Update the incumbent $\tilde{z} = 9$.

The linear relaxation of $P^{(8)}$ is unfeasible so that we pose $U^{(8)} = -\infty$.

Both $P^{(7)}$ e $P^{(8)}$ can be closed and the list is empty. The procedure above can be represented by the *branching tree* as in the picture 10.6.

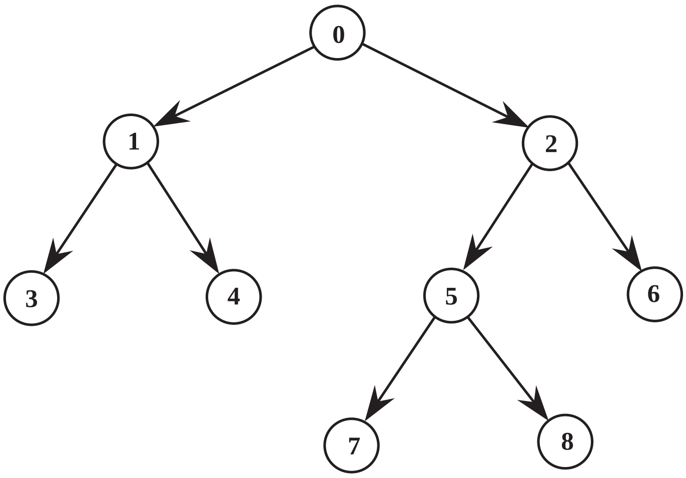


Figure 10.6: Albero di enumerazione.

## 10.13 The binary Knapsack problem

Consider the binary knapsack problem

$$\begin{aligned} \max \quad & {}^T cx \\ & {}^T ax \le b \\ & x_i \in \{0,1\}, \qquad i = 1,\ldots,n \end{aligned}$$

where $a \in \Re^n$ e $c \in \Re^n$ are positive vectors and $b > 0$. The solution of the relaxation

$$\begin{aligned} \max \quad & {}^T cx \\ & {}^T ax \le b \\ & 0 \le x_i \le 1, \qquad i = 1,\ldots,n \end{aligned}$$

can be obtained easily with the following procedure.

1. Decreasing order of the ratios:

$$\frac{c_1}{a_1} \ge \frac{c_2}{a_2} \ge \cdots \ge \frac{c_n}{a_n}$$

2. Find $h \in \{1, \ldots, n\}$ such that:

$$\begin{aligned} \sum_{i=1}^{h} a_i &\leq b \\ \sum_{i=1}^{h+1} a_i &> b \end{aligned}$$

(set $h = 0$ if $a_1 > b$).

3. The optimal solution $\bar{x}$ of the relaxation of the binary knapsack problem is

$$\begin{aligned} &\bar{x}_1 = \cdots = \bar{x}_h = 1 \\ &\bar{x}_{h+2} = \cdots = \bar{x}_n = 0 \\ &\bar{x}_{h+1} = \frac{(b - \sum_{i=1}^{h} a_i)}{a_{h+1}} \end{aligned}$$

**Example 10.14** *Consider*

$$(P_0) \qquad \begin{aligned} &\max 4x_1 + 4x_2 + 3x_3 + x_4 \\ &2x_1 + 3x_2 + 3x_3 + 2x_4 \leq 7 \\ &x \in \{0,1\}^n \end{aligned}$$

*Variables are ordered in decreasing order of the ratio* $\frac{c_i}{a_i}$

**1. Initializtion** Solve the relaxed problem to get $x_R^0 = (1, 1, \frac{2}{3}, 0)^T$ and the corresponding upper bound $U^0 = 10$. A feasible solution is easily obtained by flooring $\tilde{x} = (1, 1, 0, 0)^T$. The corresponding incumbent $\tilde{z} = 8$. We branch w.r.t. $x_3^0$ by fixing to 1 and 0 obtaining the subproblems:

$$(P_1) \qquad \begin{aligned} &\max 4x_1 + 4x_2 + 3x_3 + x_4 \\ &2x_1 + 3x_2 + 2x_4 \leq 4 \\ &x \in \{0,1\}^n, \quad x_3 = 1 \end{aligned}$$

$$(P_2) \qquad \begin{aligned} &\max 4x_1 + 4x_2 + 3x_3 + x_4 \\ &2x_1 + 3x_2 + 2x_4 \leq 7 \\ &x \in \{0,1\}^n, \quad x_3 = 0 \end{aligned}$$

We can find the optimal solution of the relaxed problems

$$x_R^1 = (1, \frac{2}{3}, 1, 0)^T, \qquad x_R^2 = (1, 1, 0, 1)^T$$

with corresponding 'upper bounds' $U^1 = 9.\overline{6}$ ed $U^2 = 9$. The list of open problem is $\mathscr{L} = \{P_1, P_2\}$.

**Step 1.**

*Closing problems dominated by the incumbent* Since $U^1 > \tilde{z}$ and $U^2 > \tilde{z}$ *none*of the two subproblems can be closed.

*Choose the subproblem from the list* We use a LIFO strategy and choose $P_2$.

*(1c) Analysis of the problem*

Since $x_R^2 \in S_2$ (it is integer) update the incumbent $\tilde{z} = U^2 = 9$ and $\tilde{x} = (1,1,0,1)^T$. $P_2$ is closed and the list is $\mathscr{L} = \{S_1\}$.

**Step 2.**

*Closing problems dominated by the incumbent.*
Since $U^1 > \tilde{z}$ problem $P^1$ cannot be closed.

*(Choose the subproblem from the list.* trivially $P_1$.

*Analysis of the problem*

Since $x_R^1 \notin S_1$ problem $P_1$ must be partitioned. We fix $x_2$ to get:

$$
\begin{array}{ll}
(P_3) & \max 4x_1 + 4x_2 + 3x_3 + x_4 \\
 & 2x_1 + 2x_4 \leq 4 \\
 & x \in \{0,1\}^n, x_3 = 1, x_2 = 0
\end{array}
$$

$$
\begin{array}{ll}
(P_4) & \max 4x_1 + 4x_2 + 3x_3 + x_4 \\
 & 2x_1 + 2x_4 \leq 1 \\
 & x \in \{0,1\}^n, x_3 = 1, x_2 = 1
\end{array}
$$

The optinal solution of the relaxed problems are respectively:

$$x_R^3 = (1,0,1,1)^T, \qquad x_R^4 = (\frac{1}{2},1,1,0)^T$$

with corresponding "upper bound" $U^3 = 8$ ed $U^4 = 9$. The list is $\mathscr{L} = \{P_3, P_4\}$.

**Step 3.**

*(Closing problems dominated by the incumbent.*
Since $U^i \leq \tilde{z}$ for both $i = 3, 4$ both subproblems can be closed. the optmal solution is the incumbent $\tilde{x} = [1,1,0,1]$ with value $\tilde{z} = 9$.