A blending model with advertising

A firm manufactures three types of gasoline (gas 1, gas 2, gas 3). These products are produced by blending the three crude oils together (crude 1, crude 2, crude 3). Each barrel of crude oil The sales price per barrel of gasoline and the purchase price per barrel of crude oil are the following

	sales price per barrel		purchase price per barrel
Gas 1	\$70	Crude 1	\$45
Gas 2	\$60	Crude 2	\$35
Gas 3	\$50	Crude 3	\$25

The firm can purchase up to 5000 barrel of each type of crude oil.

The three type of gasoline differ in their octane rating and sulfur content. The requirements are:

	at least octane rating	at most sulfur content
Gas 1	10	1%
Gas 2	8	2%
Gas 3	6	1%

The octane rating and sulfur content of the three types of crude oil are in the following table

	octane rating	sulfur content
Crude 1	12	0.5%
Crude 2	6	2.0%
Crude 3	8	3.0%

It costs €4 to transform one barrel of crude oil into one barrel of gasoline and the firm's refinery can produce up to 14,000 barrels gasoline daily. Firm's customers require the following amounts of each gasoline. gas 1- 3000 barrels per day, gas 2- 2000 barrels per day, and gas 3- 1000 barrels per day. The company considers an obligation to meet these demands.

1. Write a LP problem to plan production so to maximize the profit.
2. Sunco has also the option of advertising to stimulate demand for its product. Each dollar spent daily in advertising a particular type of gas increases demand of that type of gas by 10 barrels. For example, if Sunco decides to spend €20 daily in advertising gas 2, the daily demand for gas 2 will increase by $20(10) = 200$ barrels. Formulate an LP that will enable Sunco to maximize daily profits (profits= revenue - costs). Write a new LP problem to plan production so to maximize the profit.
3. Assume that the firm wants to advertise only one product. Add this condition to the model.

Summary of problem (an efficient tool to assist formulation)

Sulfur	Octane	Sales price	Meet demand	GAS	Decision variables			
At most	At least	€/barrel	Barrel/day					
1%	10	70	3000	1	x_{11}	x_{12}	x_{13}	A_1
2%	8	60	2000	2	x_{21}	x_{22}	x_{23}	A_2
1%	6	50	1000	3	x_{31}	x_{32}	x_{33}	A_3
				OIL	1	2	3	
				Purchase price €/barrel	45	35	25	
				Available oil, max barrel/day	5000	5000	5000	
				Octane rating	12	6	8	
				Sulfur content	0.5%	2.0%	3.0%	

* for each type of gas: 1 €ad/day will increase demand by 10 barrel/day

* €4 to transform 1 barrel oil into one barrel gas

* Max gas to produce 14000 barrel /day

Parameters:
per barrel of gasoline and the

- r_i sales price of gas i
- p_i production cost per barrel of gas i
- d_i demand for gas i
- c_j purchase price of crude oil $j-$
- a_j availability of crude oil $j-$
- P production capacity
- o_j octane rating of crude oil j
- OR_j minimum octane rating of gas i
- s_j sulfure percentage of crude oil j
- $\%S_j$ maximum sulfure percentage of gas i

QUESTION 1 Decision variables: x_{ij} = barrel of crude oil j used daily to produce gas i (i = 1, 2, 3; j = 1, 2, 3)

$$x_{ij} \geq 0$$

Useful quantities

- Used $\text{Oil}_j = \sum_{i=1}^{3} x_{ij}$ for $j = 1, 2, 3$
- Produced $\text{Gas}_i = \sum_{j=1}^{3} x_{ij}$ for $i = 1, 2, 3$

Objective function: max profit
$\max Z$ = profit = $\sum_{\#gas}$ (revenue - production cost)$_{gas_i}$ * Gas$_i$ - $\sum_{\#oils}$ (purchase cost)$_{oil_j}$ * Oil$_j$.
We can assume that the revenue is obtained by satisfying the full demand (production exceeding the demand does not produce any revenue)

$$revenue = \sum_{i=1}^{3} r_i d_i$$

$$costs = \sum_{i=1}^{3} p_i \cdot \sum_{j=1}^{3} x_{ij} + \sum_{j=1}^{3} c_j \cdot \sum_{i=1}^{3} x_{ij}$$

Constraints for Gas (1, 2, and 3) demand.

We assume that the production can exceed the demand (this may help in satisfying the quality constraints)

$$= \sum_{j=1}^{3} x_{ij} \geq \ d_i \quad \text{for any} \quad i = 1, 2, 3$$

that corresponds to

$$C1: \quad 1x_{11} + 1x_{21} + 1x_{31} \geq 3000$$
$$C2: \quad 1x_{12} + 1x_{22} + 1x_{32} \geq 2000$$
$$C3: \quad 1x_{13} + 1x_{23} + 1x_{33} \geq 1000$$

Constraints for Oil (1, 2, and 3) availability to be purchased:

$$= \sum_{i=1}^{3} x_{ij} \leq a_j \quad \text{for any} \quad j = 1, 2, 3$$

that corresponds to

$$C4: \quad 1x_{11} + 1x_{12} + 1x_{13} \leq 5000$$
$$C5: \quad 1x_{21} + 1x_{22} + 1x_{23} \leq 5000$$
$$C6: \quad 1x_{31} + 1x_{32} + 1x_{33} \leq 5000$$

Constraint on refinery capacity limit:

$$\sum_{i=1}^{3} \sum_{j=1}^{3} x_{ij} \leq P$$

that is

$$C7: \quad 1x_{11} + 1x_{12} + 1x_{13} + 1x_{21} + 1x_{22} + 1x_{23} + 1x_{31} + 1x_{32} + 1x_{33} \leq 14000$$

Constraints for minimum average required Octane level in Gas (1, 2, and 3):

$$\frac{\sum_{i=1}^{3} o_i x_{ij} \leq}{(\sum_{i=1}^{3} x_{ij}} >= OR_j \quad \text{for any} \quad j = 1, 2, 3$$

This constraints are non linear but can be easily written as linear as

$$\sum_{i=1}^{3} o_i x_{ij} \leq >= OR_j \sum_{i=1}^{3} x_{ij} \quad \text{for any} \quad j = 1, 2, 3$$

that corresponds to

$$C8: \quad (12x_{11} + 6x_{21} + 8x_{31}) - 10 \cdot (x_{11} + x_{21} + x_{31}) \geq 0$$
$$C9: \quad 4x_{12} - 2x_{22} \geq 0$$
$$C10: \quad 6x_{13} + 2x_{33} \geq 0$$

Constraints for maximum sulfur% content in Gas (1, 2, and 3):

Analogously to the case of octane rating we have

$$\sum_{i=1}^{3} s_i x_{ij} \leq>= \%S_j \sum_{i=1}^{3} x_{ij} \quad \text{for any} \quad j = 1, 2, 3$$

which are

$$\begin{array}{ll} C11: & -0.005x_{11} + 0.01x_{21} + 0.02x_{31} \leq 0 \\ C12: & -0.015x_{12} + 0.01x_{32} \leq 0 \\ C13: & -0.005x_{13} + 0.01x_{23} + 0.02x_{33} \leq 0 \end{array}$$

The full model is

$$\begin{array}{ll} \max & \sum_{i=1}^{3} r_i d_i - \left(\sum_{i=1}^{3} p_i \cdot \sum_{j=1}^{3} x_{ij} + \sum_{j=1}^{3} c_j \cdot \sum_{i=1}^{3} x_{ij} \right) \\ & \sum_{j=1}^{3} x_{ij} \geq \ d_i \quad \text{for all} \quad i = 1, 2, 3 \\ & \sum_{i=1}^{3} x_{ij} \leq a_j \quad \text{for all} \quad j = 1, 2, 3 \\ & \sum_{i=1}^{3} \sum_{j=1}^{3} x_{ij} \leq P \\ & \sum_{i=1}^{3} o_i x_{ij} \leq>= OR_j \sum_{i=1}^{3} x_{ij} \quad \text{for all} \quad j = 1, 2, 3 \\ & \sum_{i=1}^{3} s_i x_{ij} \leq>= \%S_j \sum_{i=1}^{3} x_{ij} \quad \text{for all} \quad j = 1, 2, 3 \\ & x_{ij} \geq 0, \quad i = 1, 2, 3 \quad j = 1, 2, 3 \end{array}$$

namely

$$\begin{cases} \max \ 21x_{11} + 11x_{12} + 1x_{13} + 31x_{21} + 21x_{22} + 11x_{23} + 41x_{31} + 31x_{32} + 21x_{33} \\ x_{11} + x_{21} + x_{31} \geq 3000 \\ x_{12} + x_{22} + x_{32} \geq 2000 \\ x_{13} + x_{23} + x_{33} \geq 1000 \\ x_{11} + x_{12} + x_{13} \leq 5000 \\ x_{21} + x_{22} + x_{23} \leq 5000 \\ x_{31} + x_{32} + x_{33} \leq 5000 \\ x_{11} + x_{12} + x_{13} + x_{21} + x_{22} + x_{23} + x_{31} + x_{32} + x_{33} \leq 14000 \\ 2x_{11} - 4x_{21} - 2x_{31} \geq 0 \\ 4x_{12} - 2x_{22} \geq 0 \\ 6x_{13} + 2x_{33} \geq 0 \\ -0.005x_{11} + 0.01x_{21} + 0.02x_{31} \leq 0 \\ -0.015x_{12} + 0.01x_{32} \leq 0 \\ -0.005x_{13} + 0.01x_{23} + 0.02x_{33} \leq 0 \\ x_{ij} \geq 0, \quad i = 1, 2, 3 \quad j = 1, 2, 3 \end{cases}$$

QUESTION 2

Decision variables: we introduce additional variables A_i = dollars spent daily on advertising gas i ($i = 1, 2, 3$)

These variables affects the demand that becomes

$$d_i + 10A_i$$

The demand enters both the objective function and the constraints on the demand. The rest of the model does not change.

– *Objective function.* Since the demand is increasing the revenue becomes

$$revenue = \sum_{i=1}^{3} r_i(d_i + 10A_i)$$

As for the costs we add the advertising costs

$$costs = \sum_{i=1}^{3} p_i \cdot \sum_{j=1}^{3} x_{ij} + \sum_{j=1}^{3} c_j \cdot \sum_{i=1}^{3} x_{ij} + \sum_{i=1}^{3} A_i$$

Constraints for Gas (1, 2, and 3) demand:

$$\sum_{i=1}^{3} x_{ij} \geq d_i + 10A_i \qquad i = 1, 2, 3$$

The full model becomes

$$\begin{array}{ll} \max & \displaystyle\sum_{i=1}^{3} r_i(d_i + 10A_i) - \left(\sum_{i=1}^{3} p_i \cdot \sum_{j=1}^{3} x_{ij} + \sum_{j=1}^{3} c_j \cdot \sum_{i=1}^{3} x_{ij} + \sum_{i=1}^{3} A_i \right) \\ & \displaystyle\sum_{j=1}^{3} x_{ij} - 10A_i \geq \ d_i \quad \text{for all} \ \ i = 1, 2, 3 \\ & \displaystyle\sum_{i=1}^{3} x_{ij} \leq a_j \quad \text{for all} \ \ j = 1, 2, 3 \\ & \displaystyle\sum_{i=1}^{3} \sum_{j=1}^{3} x_{ij} \leq P \\ & \displaystyle\sum_{i=1}^{3} o_i x_{ij} \leq >= OR_j \sum_{i=1}^{3} x_{ij} \quad \text{for all} \ \ j = 1, 2, 3 \\ & \displaystyle\sum_{i=1}^{3} s_i x_{ij} \leq >= \%S_j \sum_{i=1}^{3} x_{ij} \quad \text{for all} \ \ j = 1, 2, 3 \\ & x_{ij} \geq 0, \quad i = 1, 2, 3 \ \ j = 1, 2, 3 \\ & A_i \geq 0, \quad i = 1, 2, 3 \end{array}$$

that is

$$\begin{cases} \max \ 21x_{11} + 11x_{12} + 1x_{13} + 31x_{21} + 21x_{22} + 11x_{23} + 41x_{31} + 31x_{32} + 21x_{33} - A_1 - A_2 - A_3 \\ x_{11} + x_{21} + x_{31} - 10A_1 \geq 3000 \\ x_{12} + x_{22} + x_{32} - 10A_2 \geq 2000 \\ x_{13} + x_{23} + x_{33} - 10A_3 \geq 1000 \\ x_{11} + x_{12} + x_{13} \leq 5000 \\ x_{21} + x_{22} + x_{23} \leq 5000 \\ x_{31} + x_{32} + x_{33} \leq 5000 \\ x_{11} + x_{12} + x_{13} + x_{21} + x_{22} + x_{23} + x_{31} + x_{32} + x_{33} \leq 14000 \\ 2x_{11} - 4x_{21} - 2x_{31} \geq 0 \\ 4x_{12} - 2x_{22} \geq 0 \\ 6x_{13} + 2x_{33} \geq 0 \\ -0.005x_{11} + 0.01x_{21} + 0.02x_{31} \leq 0 \\ -0.015x_{12} + 0.01x_{32} \leq 0 \\ -0.005x_{13} + 0.01x_{23} + 0.02x_{33} \leq 0 \\ x_{ij} \geq 0, \quad i = 1, 2, 3 \ \ j = 1, 2, 3 \\ A_j \geq 0, \quad j = 1, 2, 3 \end{cases}$$

QUESTION 3 We introduce boolean variables

$$\delta_j = \begin{cases} 1 & \text{if investment } A_j > 0 \\ 0 & \text{otherwise} \end{cases}$$

with constraints

$$A_j \leq M\delta_j$$

where $M > 0$ is a large value. We can choose M by simple considerations. Indeed we have that

$$-10A_i \leq \sum_{j=1}^{3} x_{ij} - d_i \leq \; P - d_i \leq \; P - \min_i d_i = 13000$$

so that we can choose $M = 1300$. To select only one investment we write the constraint

$$\sum_{j=1}^{3} \delta_j = 1$$

The full model is

$$\begin{aligned}
\max \quad & \sum_{i=1}^{3} r_i(d_i + 10A_i) - \left(\sum_{i=1}^{3} p_i \cdot \sum_{j=1}^{3} x_{ij} + \sum_{j=1}^{3} c_j \cdot \sum_{i=1}^{3} x_{ij} + \sum_{i=1}^{3} A_i \right) \\
& \sum_{j=1}^{3} x_{ij} - 10A_i \geq \; d_i \quad \text{for all} \;\; i = 1, 2, 3 \\
& \sum_{i=1}^{3} x_{ij} \leq a_j \quad \text{for all} \;\; j = 1, 2, 3 \\
& \sum_{i=1}^{3} \sum_{j=1}^{3} x_{ij} \leq P \\
& \sum_{i=1}^{3} o_i x_{ij} \leq >= OR_j \sum_{i=1}^{3} x_{ij} \quad \text{for all} \;\; j = 1, 2, 3 \\
& \sum_{i=1}^{3} s_i x_{ij} \leq >= \%S_j \sum_{i=1}^{3} x_{ij} \quad \text{for all} \;\; j = 1, 2, 3 \\
& A_j \leq M\delta_j \quad j = 1, 2, 3 \\
& \sum_{i=1}^{3} \delta_i = 1 \\
& x_{ij} \geq 0, \quad i = 1, 2, 3 \;\; j = 1, 2, 3 \\
& A_i \geq 0, \quad i = 1, 2, 3 \\
& \delta_i \in \{0, 1\}, \quad i = 1, 2, 3
\end{aligned}$$

$$
\begin{cases}
\max\ 21x_{11} + 11x_{12} + 1x_{13} + 31x_{21} + 21x_{22} + 11x_{23} + 41x_{31} + 31x_{32} + 21x_{33} - A_1 - A_2 - A_3 \\
x_{11} + x_{21} + x_{31} - 10A_1 \geq 3000 \\
x_{12} + x_{22} + x_{32} - 10A_2 \geq 2000 \\
x_{13} + x_{23} + x_{33} - 10A_3 \geq 1000 \\
x_{11} + x_{12} + x_{13} \leq 5000 \\
x_{21} + x_{22} + x_{23} \leq 5000 \\
x_{31} + x_{32} + x_{33} \leq 5000 \\
x_{11} + x_{12} + x_{13} + x_{21} + x_{22} + x_{23} + x_{31} + x_{32} + x_{33} \leq 14000 \\
2x_{11} - 4x_{21} - 2x_{31} \geq 0 \\
4x_{12} - 2x_{22} \geq 0 \\
6x_{13} + 2x_{33} \geq 0 \\
-0.005x_{11} + 0.01x_{21} + 0.02x_{31} \leq 0 \\
-0.015x_{12} + 0.01x_{32} \leq 0 \\
-0.005x_{13} + 0.01x_{23} + 0.02x_{33} \leq 0 \\
A_j \leq M\delta_j, \quad j = 1,2,3 \\
\sum_{j=1}^{3} \delta_j = 1 \\
x_{ij} \geq 0, \quad i = 1,2,3 \quad j = 1,2,3 \\
A_j \geq 0, \quad j = 1,2,3 \\
\delta_j \in \{0,1\}, \quad j = 1,2,3
\end{cases}
$$