Chapter 8

Theory of Linear Programming

In this chapter the results of previous chapters will be applied to Linear programming problems (LP).
We have already seen that the feasible set of a linear programming problem is a polyhedron and therefore a convex set and that the LP is a problem that both concave and convex. Hence the solution (not necessarily unique), if it exists, is on the boundary of the feasible set.

We study some properties of polyhedrons making it possible to characterize the optimal solutions of a linear programming problem.

8.1 Vertex of a polyhedron

In this section a characterization of vertex of a polyhedron in the form $Ax \geq b$.

Theorem 8.2 (Vertex of a polyhedron) *Consider a polyhedron $S = \{x \in R^n : \ Ax \geq b\}$. A point $\bar{x} \in S$ is a vertex if and only if there are n linearly independent rows a_i^T of the matrix A corresponding to active constraints in $\bar{x}$, i.e. if and only if it*

$$rank\{A_{I(\bar{x})}\} = n$$

where $A_{I(\bar{x})}$ is the matrix $|I(\bar{x})| \times n$ made of rows of A corresponding to index in $I(\bar{x})$

Proof. We first prove the necessary part, that if $\bar{x}$ is a vertex, then $\text{rank}\{a_i : \ i \in I(\bar{x})\} = n$.

By contradiction assume that the rank is $p < n$. The homogeneous system

$$a_i^T d = 0 \quad i \in I(\bar{x})$$

in $I(\bar{x})$ equations and n unknowns, has rank less than n and thus it admits a non zero solution. We know that d is a particular feasible direction. Furthermore we note that, since d is a solution of the homogeneous system solution, also $-d$ is a feasible direction. Then we can consider the two points

$$y = \bar{x} + td$$
$$z = \bar{x} + t(-d)$$

and we know that for values of t sufficiently small are both feasible. Now we can write

$$\bar{x} = \frac{1}{2}y + \frac{1}{2}z$$

i.e. $\bar{x}$ can be obtained as a convex combination with coefficient $\beta = \frac{1}{2}$ of two feasible and distinct points. But this contradicts that $\bar{x}$ is a vertex.

We prove now the sufficient part, namely that if $\text{rank}\{a_i : \ i \in I(\bar{x})\} = n$ then $\bar{x}$ is a vertex. We first observe that the condition implies that $\bar{x}$ is the only solution to the system

$$a_i^T x = b_i \ \text{ for } \ i \in I(\bar{x}).$$

Now proceed by contradiction and suppose that the point $\bar{x}$ is not a vertex. Then it cannot be the only feasible point and in particular exist two feasible points v and w different from $\bar{x}$ and such that $\bar{x}$ can be expressed as a convex combination of v and w i.e.

$$\bar{x} = (1-\beta)v + \beta w \qquad \text{con } \beta \in (0,1).$$

For each $i \in I(\bar{x})$ we can write

$$b_i = a_i^T \bar{x} = (1-\beta)a_i^T v + \beta a_i^T w$$

We note that it must necessarily be $a_i^T v = b_i$. Indeed, if it were not true, and either $a_i^T v > b_i$ or $a_i^T w > b_i$ you would have the absurd

$$b_i = a_i^T \bar{x} = (1-\beta)a_i^T v + \beta a_i^T w > (1-\beta)b_i + \beta b_i = b_i.$$

But then we get that both v and w are solutions of the system

$$a_i^T x = b_i \ \text{ for } \ i \in I(\bar{x})$$

which contradicts the assumption. □

Corollary 8.3 *Let a polyhedron $S = \{x \in \Re^n : \ Ax \geq b\}$. If the matrix A has a number of rows linearly independent less than n, then S has no vertices.*
Particularly if $m < n$ then S has no vertices.

Corollary 8.4 *A polyhedron $S = \{x \in \Re^n : \ Ax \geq b\}$ has a finite number of vertexes, amounting to a maximum*

$$\begin{pmatrix} m \\ n \end{pmatrix} = \frac{m!}{n!(m-n)!}$$

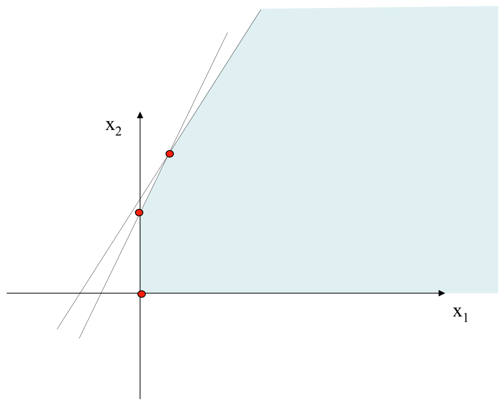

Figure 8.1: Polyhedron of Exercise 8.5.

Example 8.5 Consider the polyhedron

$$\begin{array}{l} 3x_1 - 2x_2 \geq -30 \\ 2x_1 - x_2 \geq -12 \\ x_1 \geq 0 \; x_2 \geq 0 \end{array}$$

which is

$$\begin{pmatrix} -3 & -2 \\ 2 & -1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \geq \begin{pmatrix} -30 \\ -12 \\ 0 \\ 0 \end{pmatrix}.$$

The polyhedron is shown in Figure 8.5 and the vertices are marked with a red dot. We verify that the condition expressed by Theorem 8.2 are verified. The three vertices are the points

$$v_1 = \begin{pmatrix} 0 \\ 12 \end{pmatrix} \quad v_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad v_3 = \begin{pmatrix} 6 \\ 24 \end{pmatrix}.$$

In v_1 we have as active constraint $2x_1 - x_2 \geq -12, \; x_1 \geq 0$, i.e $I(v_1) = \{2,3\}$ and the matrix in the theorem is

$$\begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}$$

which are linearly independent.
Check for the other points.

We note that theorem 8.2 does not exclude that in a vertex more than n constraints are active.

Example 8.6 Consider again Example 8.5 with and additional constraint $x_2 \leq 24$, namely:

$$\begin{array}{l} 3x_1 - 2x_2 \geq -30 \\ 2x_1 - x_2 \geq -12 \\ -x_2 \geq -24 \\ x_1 \geq 0, \\ x_2 \geq 0 \end{array}$$

The point $v_3 = \begin{pmatrix} 6 \\ 24 \end{pmatrix}$ is a vrtex but in v_3 the actve constraints are three $I(v_3) = \{1,2,3\}$. The matrix in the theorem is

$$\begin{pmatrix} 3 & -2 \\ 2 & -1 \\ 0 & -1 \end{pmatrix}$$

which has rak equal to two. □

However you have to pay attention to the fact that there are polyhedron *that do not contain* vertices. An example is given in figure 8.2 where the polyhedron is the part of plan contained between two parallel lines r_1 and r_2.

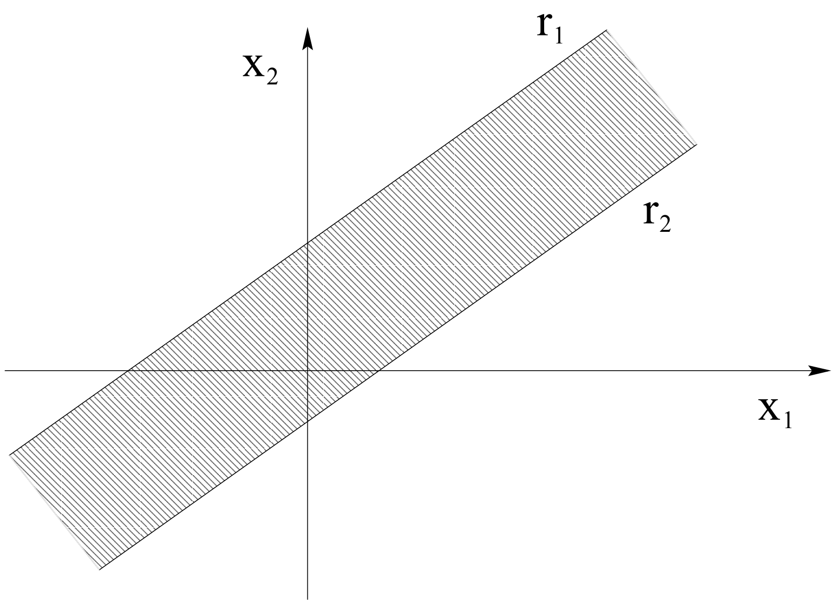

Figure 8.2: polyhedron without verteces.

Definition 8.7 (Line) *We say that a polyhedron S contains a line if there exists a point $\bar{x} \in S$ and a direction $d \in \Re^n$ such that*

$$\bar{x} + td \in S \text{ for all } t \in \Re.$$

Theorem 8.8 *A not empty polyhedron P has no vertices if and only if it contains a line.*

Remark 8.9 *In an LP problem, if variables are constrained to be non negative, that is $x \geq 0$, the feasible polyhedron is fully contained in the positive orthant and then* it cannot contain a line.

Hence we can state

If a polyhedron

$$S \subseteq \{x \in \mathbb{R}^n : x \geq 0\}$$

is not empty, it admits at least one vertex.

In particular

Corollary 8.10 *A polyhedron $S = \{x \in \Re^n : Ax = b, x \geq 0\}$, if not empty, admits always a vertex*

Theorem 8.11 (Vertex of a polyhedron in stadard form) *Consider $S = \{x \in \Re^n : Ax = b, x \geq 0\}$. A point $\bar{x} \in S$ is a vertex if and only if the columns of the matrix A corresponding to positive components of $\bar{x}$, are linearly independent.*

We note that the polyhedron

$$S = \{x \in \mathbb{R}^n : Ax \geq b, x \geq 0\}$$

(with A $m \times n$ matrix) can be transformed into standard form with the addition of surplus variables as follows

$$S' = \left\{ \begin{pmatrix} x \\ s \end{pmatrix} \in \mathbb{R}^{n+m} : Ax - s = b, x \geq 0, s \geq 0 \right\}.$$

Theorem 8.12 *A point $\bar{x}$ is a vertex of the ployhedron $S = \{x \in \mathbb{R}^n : Ax \geq b, x \geq 0\}$ if and only if $\begin{pmatrix} \bar{x} \\ \bar{s} \end{pmatrix}$ with $\bar{s} = A\bar{x} - b$ is a vertex of the ployhedron $S' = \left\{ \begin{pmatrix} x \\ s \end{pmatrix} \in \mathbb{R}^{n+m} : Ax - s = b, x \geq 0, s \geq 0 \right\}$.*

8.13 The fundamental theorem of LP

Consider the LP

$$\begin{array}{ll} \min & c^T x \\ & Ax \geq b \end{array} \tag{8.1}$$

Theorem 8.14 (Fundamental theorem of Linear Programming) *Consider an LP problem. Then one and only one of the following statements is true:*

1. *The feasible region is empty;*

2. *the problem is unbounded;*

3. *the problem admits an optimal solution.*

If the problem admits optimal solutions and the feasible polyhedron has vertexes, then at least one optimal solution is on a vertex.

In the proof of the fundamental theorem of PL, we will refer to problems in the form of PL (8.1) and we will use the hypothesis (underline just for simplifying the demonstration) that the polyhedron does not contain a line, which, on the basis Theorem 8.8, ensures of the existence of at least one vertex of the polyhedron.

Theorem 8.15 (Fundamental theorem of Linear Programming) *Consider an LP problem of type (8.1). Assume that the polyhedron $S = \{x \in R^n : \ Ax \geq b\}$ does not contain a line. Then one and only one of the following statements is true:*

(a) *The feasible region is empty;*

(b) *the problem is unbounded;*

(c) *the problem admits an optimal solution and at least one optimal solution is on a vertex.*

Proof. Obviously the three statements are incompatible in the sense that if one is true, the other two cannot be true. So to prove the theorem, it will be enough to show that it can not happen that there is no none of the three. We will show this by proving that if neither (a) nor (b) are true, then (c) must be the true.

Suppose that the feasible region is not empty and that the problem is not unbounded below. We need to prove that there is a vertex v^* such that $c^T v^* \leq c^T x$ for all feasible x.
If v^* is the only feasible point it is also the optimal solution and vertex. Assume that we have more than one point.

The proof continues in two parts. In the first part we prove the following statement

$$\text{For any } \tilde{x} \in S, \text{ a vertex } v^k \text{ exists satisfying } c^T v^k \leq c^T \tilde{x}.$$

Then we can use known results. Indeed from Theorem 8.4, we know that vertices of the polyhedron are finite: $v^1, \dots v^p$. So among all vertices v^i we can choose the one for which the value of the objective function takes the minimum value, which we denote by v^*.

It is therefore

$$c^T v^* \leq c^T v^k$$

Putting together the two statements in the red, we can finally write

$$c^T v^* \leq c^T v^k \leq c^T \tilde{x}, \quad \text{per ogni } \tilde{x} \in S$$

which proves assertion (c).
We only need to prove the first part. Let $\tilde{x}$ ne a feasible solution which is not a vertex. By Theorem 8.2, the system

$$a_i^T d = 0, \quad i \in I(\tilde{x})$$

has no full row rank and hence it admits a non zero solution $\bar{d}$. Hence both $\pm\bar{d}$ are feasible directions and the point $\tilde{x}+t\bar{d}$ with $t \in [0, t^+_{\max}]$ and $\tilde{x}-t\bar{d}$ with $t \in [0, t^-_{\max}]$ are feasible. We show now that $\min\{t^+_{\max}, t^-_{\max}\} < \infty$, namely that the movement along one of the two direction is finite. Let us consider w.l.g the direction $\bar{d}$ such that $c^T\bar{d} \le 0$ so that we have one of the two cases:

(i) $c^T\bar{d} = 0$;

(ii) $c^T\bar{d} < 0$.

When $c^T\bar{d} = 0$ also $c^T(-\bar{d}) = 0$. Since the polyhedron does not contain a line at least one of $t^+_{\max}$ and $t^-_{\max}$ is $< \infty$. Assume wlg that $t^+_{\max} < \infty$.
If $c^T\bar{d} < 0$, the direction $\bar{d}$ is a descent direction (see section 3.1)and for all $t > 0$ we have

$$c^T(\tilde{x}+td) = c^T\tilde{x} + tc^Td = c^T\tilde{x} - t|c^Td| \quad \text{for all } t > 0.$$

Assume $t \to \infty$. We get

$$\lim_{t\to\infty} c^T(\tilde{x}+td) = c^T\tilde{x} - |c^Td| \lim_{t\to\infty} t = -\infty.$$

This cannot happen because we assumed that the problem is not unbounded below, so that $t \not\to \infty$ and $t^+_{\max} < \infty$.
In both cases we get that $\tilde{x}+t\bar{d}$ is feasible for $0 \le t \le t^+_{\max} < \infty$.
Consider the point

$$y = \tilde{x} + t^+_{\max}\bar{d};$$

since $c^T\bar{d} \le 0$ we have

$$c^Ty = c^T\tilde{x} - t^+_{\max}|c^Td| \le c^T\tilde{x}.$$

Let's check which are the active constraints in y.For $i \in I(\tilde{x})$ we have $a_i^T\bar{d} = 0$, so that

$$a_i^Ty = a_i^T(\tilde{x} + t^+_{\max}\bar{d}) = a_i^T\tilde{x} = b_i \qquad i \in I(\tilde{x})$$

Hence $I(\tilde{x}) \subseteq I(y)$. We prove that $I(\tilde{x}) \subset I(y)$, namely that in y at least one more constraint with respect to $\tilde{x}$ is activated. Consider inactive constraints in $\tilde{x}$ and let $j_{\max} \notin I(\tilde{x})$ be one index such that:

$$t^+_{\max} = \frac{a^T_{j_{\max}}\tilde{x} - b_{j_{\max}}}{|a^T_{j_{\max}}d|}$$

By definition $j_{\max} \notin I(\tilde{x})$ and $a^T_{j_{\max}}\bar{d} < 0$. We get

$$a^T_{j_{\max}}x = a^T_{j_{\max}}\tilde{x} + t^+_{\max}a^T_{j_{\max}}\bar{d} = a^T_{j_{\max}}\tilde{x} - t^+_{\max}|a^T_{j_{\max}}d| = a^T_{j_{\max}}\tilde{x} - \frac{a^T_{j_{\max}}\tilde{x} - b_{j_{\max}}}{|a^T_{j_{\max}}\bar{d}|}|a^T_{j_{\max}}\bar{d}| = b_{j_{\max}}$$

hence $I(y) \supseteq I(\tilde{x}) \cup \{j_{\max}\}$.

We then demonstrated that, starting from any point feasible $\tilde{x}$, we can determine a new point y with value of objective function not higher and with a number of linearly independent active constraints greater than $\tilde{x}$. If y is not a vertex, we can repeat the same process until we find a point where n linearly independent constraints are active, that is a vertex.

Hence we have finally proved the statement

For any feasible $x \in S$, a vertex v exists such that $c^T v \leq c^T x$.

Theorem 8.16 (Fundamental theorem of Linear Programming) *Consider the LP*

$$\begin{aligned} \min \quad & c^T x \\ & Ax = b \\ & x \geq 0. \end{aligned}$$

Then one and only one of the following statements is true:

(a) *The feasible region is empty;*

(b) *the problem is unbounded;*

(c) *the problem admits an optimal solution and at least one optimal solution is on a vertex.*